

Invertible bimodule categories over the representation category of a Hopf algebra

Bojana Femić ^a, Adriana Mejía Castaño ^b and Martín Mombelli ^b

^aFacultad de Ingeniería,
Universidad de la República
Julio Herrera y Reissig 565, 11 300 Montevideo, Uruguay

^bCIEM-FAMAF,
Universidad Nacional de Córdoba
Medina Allende s/n, 5000, Córdoba, Argentina

February 13, 2014

Abstract

For any finite-dimensional Hopf algebra H we construct a group homomorphism $\text{BiGal}(H) \rightarrow \text{BrPic}(\text{Rep}(H))$, from the group of equivalence classes of H -biGalois objects to the group of equivalence classes of invertible exact $\text{Rep}(H)$ -bimodule categories. We discuss the injectivity of this map. We exemplify in the case $H = T_q$ is a Taft Hopf algebra and for this we classify all exact indecomposable $\text{Rep}(T_q)$ -bimodule categories.

Mathematics Subject Classification (2010): 18D10, 16W30, 19D23.

Keywords: Brauer-Picard group, tensor category, biGalois object.

1 Introduction

The Brauer-Picard group $\text{BrPic}(\mathcal{C})$ of a finite tensor category \mathcal{C} is the group of equivalence classes of invertible exact \mathcal{C} -bimodule categories. This group, and its higher versions, were introduced in [10] to classify extensions of a given tensor category \mathcal{C} by a finite group. Also, it has a close relation to certain structures appearing in mathematical physics, like rational Conformal Field Theory or 3-dimensional Topological Field Theory, see for example [12], [9], [16].

In order to classify extensions of a tensor category \mathcal{C} by a finite group G one needs a group map $G \rightarrow \text{BrPic}(\mathcal{C})$ and certain cohomological data, [11]. Henceforth, determining any subgroup of $\text{BrPic}(\mathcal{C})$ presents a significant step in the mentioned classification.

The Brauer-Picard group is a complicated group to compute even in the simplest examples. One interesting problem is the computation of $\text{BrPic}(\text{Rep}(H))$, where H is a finite-dimensional Hopf algebra. It is known that any exact $\text{Rep}(H)$ -bimodule category is equivalent to the category of finite-dimensional representations of a left $H \otimes H^{\text{cop}}$ -comodule algebra. Given a left $H \otimes H^{\text{cop}}$ -comodule algebra the problem of deciding when its category of representations is invertible is not solved. The main difficulty is that the Deligne's tensor product of bimodule categories is not easy to compute explicitly.

One of the principle goals of this paper is the description of a certain family of invertible exact $\text{Rep}(H)$ -bimodule categories coming from H -biGalois objects. They give rise to a subgroup of $\text{BrPic}(\text{Rep}(H))$. This result is expressed in Corollary 4.9. Here we construct a short exact sequence involving a map from the group of isomorphism classes of H -biGalois objects $\text{BiGal}(H)$ to $\text{BrPic}(\text{Rep}(H))$ and its kernel. As a consequence we get that for any co-quasitriangular Hopf algebra H the group $\text{BiGal}(H)$ embeds into the Brauer-Picard group $\text{BrPic}(\text{Rep}(H))$.

The subsequent part of the paper (the Section 5) is dedicated to the study of the case when H is the Taft Hopf algebra $T_q = \mathbb{k}\langle g, x | g^n = 1, x^n = 0, gx = qxg \rangle$, where q is a primitive n -th root of unity. Our second main goal is to classify all exact indecomposable $\text{Rep}(T_q)$ -bimodule categories; we obtain five families of them. As announced, some families of invertible bimodule categories arise from biGalois objects.

The approach we use to classify the exact indecomposable $\text{Rep}(T_q)$ -bimodule categories is the following. Let H be a finite-dimensional Hopf algebra. Any exact indecomposable $\text{Rep}(H)$ -bimodule category is equivalent to the category of finite-dimensional representations of a left $H \otimes H^{\text{cop}}$ -comodule algebra which is $H \otimes H^{\text{cop}}$ -simple (it has no non-trivial ideals which are simultaneously left $H \otimes H^{\text{cop}}$ -comodules) and with trivial coinvariants. So it is enough to find all comodule algebras over $H \otimes H^{\text{cop}}$ with these properties. By [23] any coideal subalgebra A of H is H -simple, and due to [18, Remark 3.2] the representation category of A twisted by a *compatible* 2-cocycle over H is an exact indecomposable $\text{Rep}(H)$ -bimodule category. Moreover, the *liftings* of cocycle twisted coideal subalgebras are as well H -simple comodule algebras with trivial coinvariants. Here, a *lifting* of a coideal subalgebra K is a comodule algebra A such that $\text{gr } A \simeq K$. Then setting $H = T_q \otimes T_q^{\text{cop}}$, we determine all homogeneous coideal subalgebras of H and their 2-cocycle twists. We determine the liftings of them and get five families of H -comodule algebras. By the above, the representation categories of these five families are exact indecomposable $\text{Rep}(T_q)$ -bimodule categories. In Theorem 5.17 we prove that every exact indecomposable $\text{Rep}(T_q)$ -bimodule category is of this form. This classification result is interesting in itself. The biGalois objects over T_q arise from one of the five families of the above comodule algebras and from their form it is straightforward that the kernel of the map $\text{BiGal}(T_q) \rightarrow \text{BrPic}(\text{Rep}(T_q))$ is trivial. Hence, the group $\text{BiGal}(T_q)$ embeds into $\text{BrPic}(\text{Rep}(T_q))$, although T_q is not co-quasitriangular.

The contents of the paper are the following. In Section 2 we give the necessary preliminaries on tensor categories and their representations. We also prove that for Galois objects over Hopf algebras, the tensor product of bimodule categories can be given in an explicit form. In Section 3 we recall some basic notions on bicategories that we use later. We also give a proof of a known result; that the bicategory of representations of a given tensor category determines the tensor category up to Morita equivalence. In Section 4 we present a group homomorphism $\text{BiGal}(H) \rightarrow \text{BrPic}(\text{Rep}(H))$. The major part of this

section is dedicated to describe its kernel as well as possible. For co-quasitriangular Hopf algebras this map is always injective. In Subsection 5.1 we compute the 2-cocycle twists of $H = T_q \otimes T_q^{cop}$. In the next subsection we find all homogeneous coideal subalgebras of H . In Subsection 5.3 we introduce five families of H -comodule algebras that are H -simple and with trivial coinvariants which turn out to be liftings of the coideal subalgebras. In Subsection 5.4 we classify all exact indecomposable $\text{Rep}(T_q)$ -bimodule categories and prove that they all come from the above five families of comodule algebras. We also determine the biGalois objects over T_q . The last subsection is dedicated to the explicit embedding of $\mathbb{k}^\times \ltimes \mathbb{k}^+ \simeq \text{BiGal}(T_q)$ into $\text{BrPic}(\text{Rep}(T_q))$.

2 Preliminaries and Notation

We shall work over an algebraically closed field \mathbb{k} of characteristic zero. If G is a finite group and $\psi \in Z^2(G, \mathbb{k}^\times)$ is a 2-cocycle there is another 2-cocycle ψ' in the same cohomology class as ψ such that

$$\psi'(g, 1) = \psi'(1, g) = 1, \quad \psi'(g, g^{-1}) = 1, \quad \psi'(g, h)^{-1} = \psi'(h^{-1}, g^{-1}), \quad (1)$$

for all $g, h \in G$.

All vector spaces and algebras are assumed to be over \mathbb{k} . We denote by $\text{vect}_{\mathbb{k}}$ the category of finite-dimensional \mathbb{k} -vector spaces. If A is an algebra we shall denote by ${}_A\mathcal{M}$ the category of finite-dimensional left A -modules.

If C is a coalgebra and V is a right C -comodule with comodule structure $\rho_V : V \rightarrow V \otimes_{\mathbb{k}} C$ and W is a left C -comodule with comodule structure $\lambda_W : W \rightarrow C \otimes_{\mathbb{k}} W$, we denote by $W^{\text{co}C} = \{w \in W : \lambda_W(w) = 1 \otimes w\}$ the set of (left) coinvariants of W and $V \square_C W$ is the equalizer of the arrows $\rho_V \otimes \text{id}, \text{id} \otimes \lambda_W : V \otimes_{\mathbb{k}} W \rightarrow V \otimes_{\mathbb{k}} C \otimes_{\mathbb{k}} W$.

Let H be a finite-dimensional Hopf algebra. We denote by $G(H)$ the group of group-like elements in H . We shall denote by $\text{Rep}(H)$ the tensor category of finite dimensional left H -modules and $\text{Comod}(H)$ the tensor category of finite dimensional left H -comodules. For the basic notions and definitions of Hopf Galois extensions over a Hopf algebra the reader is referred to [22].

2.1 Hopf algebras and comodule algebras

Given a coradically graded Hopf algebra $H = \bigoplus_{i=0}^m H(i)$ we say that a left coideal subalgebra $K \subseteq H$ is *homogeneous* if it is a graded algebra $K = \bigoplus_{i=0}^m K(i)$ such that $K(i) \subseteq H(i)$.

Let H be a finite-dimensional Hopf algebra. If (K, λ) is a left H -comodule algebra we denote by \overline{K} the right H -comodule algebra with opposite underlying algebra K^{op} and coaction $\overline{\lambda} : K \rightarrow K \otimes_{\mathbb{k}} H$ given by

$$\overline{\lambda}(k) = k_{(0)} \otimes S^{-1}(k_{(1)}), \quad \text{for all } k \in K.$$

If L, K are right H -comodule algebras, we denote by ${}_L\mathcal{M}_K^H$ the category of (L, K) -bimodules with a right H -comodule structure such that it is a morphism of (L, K) -bimodules. If L, K are left H -comodule algebras the category ${}_L^H\mathcal{M}_K$ can be defined similarly.

An H -comodule algebra is said to be *H -simple* if it has no non-trivial H -costable ideals.

2.2 Twisting Hopf algebras

Let H be a Hopf algebra. Let us recall that a Hopf 2-cocycle for H is a map $\sigma : H \otimes_{\mathbb{k}} H \rightarrow \mathbb{k}$, invertible with respect to convolution, such that

$$\sigma(x_{(1)}, y_{(1)})\sigma(x_{(2)}y_{(2)}, z) = \sigma(y_{(1)}, z_{(1)})\sigma(x, y_{(2)}z_{(2)}), \quad (2)$$

$$\sigma(x, 1) = \varepsilon(x) = \sigma(1, x), \quad (3)$$

for all $x, y, z \in H$. Using this cocycle there is a new Hopf algebra structure constructed over the same coalgebra H with the product described by

$$x \cdot_{\sigma} y = \sigma(x_{(1)}, y_{(1)})\sigma^{-1}(x_{(3)}, y_{(3)})x_{(2)}y_{(2)}, \quad x, y \in H \quad (4)$$

This new Hopf algebra is denoted by $H^{[\sigma]}$. If $\sigma : H \otimes H \rightarrow \mathbb{k}$ is a Hopf 2-cocycle and A is a left H -comodule algebra, then there is a new product in A given by

$$a \cdot_{\sigma} b = \sigma(a_{(-1)}, b_{(-1)})a_{(0)} \cdot b_{(0)}, \quad (5)$$

$a, b \in A$. We shall denote by A_{σ} this new algebra. The algebra A_{σ} is a left $H^{[\sigma]}$ -comodule algebra.

Let H be a coradically graded pointed Hopf algebra with coradical $H_0 = \mathbb{k}G$. Let $\psi \in Z^2(G, \mathbb{k}^{\times})$. For the proof of the next result see [15, Lemma 4.1].

Lemma 2.1 *There exists a Hopf 2-cocycle $\sigma_{\psi} : H \otimes_{\mathbb{k}} H \rightarrow \mathbb{k}$ such that for any homogeneous elements $x, y \in H$*

$$\sigma_{\psi}(x, y) = \begin{cases} \psi(x, y), & \text{if } x, y \in H(0); \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

□

2.3 Relative Hopf modules

Let H be a finite-dimensional Hopf algebra. Let K be a left H -comodule algebra and L a right H -comodule algebra. Define the functors

$$F : {}_{L \square_H K} \mathcal{M} \rightarrow {}_L \mathcal{M}_{\overline{K}}^H, \quad G : {}_L \mathcal{M}_{\overline{K}}^H \rightarrow {}_{L \square_H K} \mathcal{M}$$

as

$$F(M) = (L \otimes_{\mathbb{k}} K) \otimes_{L \square_H K} M, \quad G(N) = N^{\text{co}H},$$

for all $M \in {}_{L \square_H K} \mathcal{M}, N \in {}_L \mathcal{M}_{\overline{K}}^H$.

Theorem 2.2 [7, Thm. 4.2] *If L is a Hopf-Galois extension then the pair of functors (F, G) gives an equivalence of categories.* □

Lemma 2.3 *There is an equivalence of categories ${}_L \mathcal{M}_{\overline{K}}^H \simeq {}_K^H \mathcal{M}_{\overline{L}}$.*

Proof. Define the functor $I : {}_L\mathcal{M}_{\overline{K}}^H \rightarrow {}_K^H\mathcal{M}_{\overline{L}}$ by $I(M) = M$. If $\delta : M \rightarrow M \otimes_{\mathbb{k}} H$, $\delta(m) = m_{(0)} \otimes m_{(1)}$, $m \in M$, is the comodule structure then the left H -comodule structure on $I(M)$ is given by

$$\widehat{\delta} : M \rightarrow H \otimes_{\mathbb{k}} M, \quad \widehat{\delta}(m) = \mathcal{S}^{-1}(m_{(1)}) \otimes m_{(0)},$$

for all $m \in M$. It is not difficult to prove that this functor is well-defined and gives an equivalence of categories. \square

2.4 Tensor categories, their representations and the Brauer-Picard group

A *tensor category* over \mathbb{k} is a \mathbb{k} -linear Abelian rigid monoidal category with \mathbb{k} -bilinear tensor product. Hereafter all tensor categories will be assumed to be over a field \mathbb{k} . A *finite category* is an Abelian \mathbb{k} -linear category such that it is equivalent to the category of finite-dimensional representations of a finite-dimensional \mathbb{k} -algebra. A *finite tensor category* [12] is a tensor category with finite underlying Abelian category such that the unit object is simple. All functors will be assumed to be \mathbb{k} -linear and all categories will be finite.

If \mathcal{C} is a tensor category, we shall denote by \mathcal{C}^{rev} the tensor category whose underlying Abelian category is \mathcal{C} and the reversed tensor product: $X \otimes^{\text{rev}} Y = Y \otimes X$, $X, Y \in \mathcal{C}$. The associativity of \mathcal{C}^{rev} is given by $a_{X,Y,Z}^{\text{rev}} = a_{Z,Y,X}^{-1}$ for $X, Y, Z \in \mathcal{C}$.

For the definition of left and right module categories over a tensor category we refer to [12]. Let \mathcal{C}, \mathcal{D} be finite tensor categories. For the definition of a $(\mathcal{C}, \mathcal{D})$ -bimodule category we refer to [16], [11]. In few words a $(\mathcal{C}, \mathcal{D})$ -bimodule category is the same as left $\mathcal{C} \boxtimes \mathcal{D}^{\text{rev}}$ -module category. Here \boxtimes denotes the Deligne tensor product of two finite abelian categories.

A $(\mathcal{C}, \mathcal{D})$ -bimodule category is *decomposable* if it is the direct sum of two non-trivial $(\mathcal{C}, \mathcal{D})$ -bimodule categories. A $(\mathcal{C}, \mathcal{D})$ -bimodule category is *indecomposable* if it is not decomposable. A $(\mathcal{C}, \mathcal{D})$ -bimodule category is *exact* if it is exact as a left $\mathcal{C} \boxtimes \mathcal{D}^{\text{rev}}$ -module category, [11], [16].

If $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ are tensor categories and \mathcal{M} is a $(\mathcal{C}_1, \mathcal{C}_2)$ -bimodule category and \mathcal{N} is a $(\mathcal{C}_2, \mathcal{C}_3)$ -bimodule category, the tensor product over \mathcal{C}_2 is denoted by $\mathcal{M} \boxtimes_{\mathcal{C}_2} \mathcal{N}$. This category is a $(\mathcal{C}_1, \mathcal{C}_3)$ -bimodule category. For more details on the tensor product of module categories the reader is referred to [11], [16].

If \mathcal{M} is a right \mathcal{C} -module category then \mathcal{M}^{op} denotes the opposite Abelian category with left \mathcal{C} action $\mathcal{C} \times \mathcal{M}^{\text{op}} \rightarrow \mathcal{M}^{\text{op}}$, $(X, M) \mapsto M \overline{\otimes} X^*$ and associativity isomorphisms $m_{X,Y,M}^{\text{op}} = m_{M,Y^*,X^*}$ for all $X, Y \in \mathcal{C}, M \in \mathcal{M}$. Similarly, if \mathcal{M} is a left \mathcal{C} -module category. If \mathcal{M} is a $(\mathcal{C}, \mathcal{D})$ -bimodule category then \mathcal{M}^{op} is a $(\mathcal{D}, \mathcal{C})$ -bimodule category. See [16, Prop. 2.15].

A $(\mathcal{C}, \mathcal{D})$ -bimodule category \mathcal{M} is called *invertible* [11] if there are equivalences of bimodule categories

$$\mathcal{M}^{\text{op}} \boxtimes_{\mathcal{C}} \mathcal{M} \simeq \mathcal{D}, \quad \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{M}^{\text{op}} \simeq \mathcal{C}.$$

Tensor categories \mathcal{C} and \mathcal{D} are said to be *Morita equivalent* if there exists an indecomposable exact left \mathcal{C} -module category \mathcal{M} and a tensor equivalence $\mathcal{D}^{\text{rev}} \simeq \text{End}_{\mathcal{C}}(\mathcal{M})$.

The following result seems to be well-known.

Lemma 2.4 *Let \mathcal{C}, \mathcal{D} be tensor categories. The following statements are equivalent.*

1. *The categories \mathcal{C} and \mathcal{D} are Morita equivalent;*
2. *there exists an invertible $(\mathcal{C}, \mathcal{D})$ -bimodule category.*

Proof. (2) implies (1) is part of [11, Proposition 4.2]. Now, let us assume that the tensor categories \mathcal{C}, \mathcal{D} are Morita equivalent. Let $\Phi : \mathcal{D}^{\text{rev}} \xrightarrow{\sim} \text{End}_{\mathcal{C}}(\mathcal{M})$ be a tensor equivalence. Since \mathcal{M} is an indecomposable exact left $\text{End}_{\mathcal{C}}(\mathcal{M})$ -module category then it is an indecomposable exact right \mathcal{D} -module category. The right \mathcal{D} -action is given as follows:

$$\mathcal{M} \times \mathcal{D} \rightarrow \mathcal{M}, \quad M \overline{\otimes} Y = \Phi(Y)(M),$$

for all $M \in \mathcal{M}, Y \in \mathcal{D}$. It is easy to prove that \mathcal{M} is an exact $(\mathcal{C}, \mathcal{D})$ -bimodule category. The functor Φ is an equivalence of $(\mathcal{D}, \mathcal{D})$ -bimodule categories. Thus \mathcal{M} is an invertible $(\mathcal{C}, \mathcal{D})$ -bimodule category. \square

Given a finite tensor category \mathcal{C} , the *Brauer-Picard group* $\text{BrPic}(\mathcal{C})$ of \mathcal{C} [11] is the group of equivalence classes of invertible exact \mathcal{C} -bimodule categories. This group does not depend on the Morita class of the tensor category; if \mathcal{D} is another tensor category Morita equivalent to \mathcal{C} there is an isomorphism $\text{BrPic}(\mathcal{C}) \simeq \text{BrPic}(\mathcal{D})$. Let us explain this isomorphism. Let \mathcal{M} be an invertible $(\mathcal{C}, \mathcal{D})$ -bimodule category. Define

$$\Phi : \text{BrPic}(\mathcal{C}) \rightarrow \text{BrPic}(\mathcal{D}), \quad \Phi([\mathcal{N}]) = [\mathcal{M}^{\text{op}} \boxtimes_{\mathcal{C}} \mathcal{N} \boxtimes_{\mathcal{C}} \mathcal{M}], \quad (7)$$

for all $[\mathcal{N}] \in \text{BrPic}(\mathcal{C})$. Here $[\mathcal{N}]$ denotes the equivalence class of the module category \mathcal{N} .

2.5 Generating some invertible bimodule categories

There is a natural way to construct invertible \mathcal{C} -bimodule categories using tensor autoequivalences. See for example [11], [14]. Let \mathcal{C}, \mathcal{D} be finite tensor categories and $(F, \xi) : \mathcal{C} \rightarrow \mathcal{C}, (G, \zeta) : \mathcal{D} \rightarrow \mathcal{D}$ be tensor autoequivalences. If \mathcal{M} is a $(\mathcal{C}, \mathcal{D})$ -bimodule category we shall denote by ${}^F\mathcal{M}^G$ the following $(\mathcal{C}, \mathcal{D})$ -bimodule category. The category ${}^F\mathcal{M}^G$ has underlying Abelian category equal to \mathcal{M} . The left and right actions are given by

$$X \overline{\otimes} M = F(X) \overline{\otimes} M, \quad M \overline{\otimes} Y = M \overline{\otimes} G(Y),$$

for all $X \in \mathcal{C}, Y \in \mathcal{D}, M \in \mathcal{M}$. The left and right associativity

$$m_{X,Y,M}^F = m_{X,Y,M}^l(\xi_{X,Y} \otimes \text{id}_M), \quad m_{M,X,Y}^G = m_{M,X,Y}^r(\text{id}_M \otimes \zeta_{X,Y}).$$

Here m^l (resp. m^r) is the left (resp. the right) associativity constraint of \mathcal{M} . If G is the identity functor we shall denote ${}^F\mathcal{M}^G$ simply by ${}^F\mathcal{M}$ and if F is the identity we shall denote ${}^F\mathcal{M}^G$ by \mathcal{M}^G . Let $\text{Aut}(\mathcal{C})$ denote the group of tensor autoequivalences of \mathcal{C} .

Lemma 2.5 *Let $F, G \in \text{Aut}(\mathcal{C})$ and let \mathcal{M} be a \mathcal{C} -bimodule category. The following statements hold.*

1. *There are equivalences of bimodule categories $\mathcal{C}^F \boxtimes_{\mathcal{C}} \mathcal{C}^G \simeq \mathcal{C}^{FG}$. In particular \mathcal{C}^F is invertible.*

2. There are equivalences of bimodule categories

$$\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{C}^F \simeq \mathcal{M}^F, \quad (\mathcal{M}^{\text{op}})^F \simeq ({}^F \mathcal{M})^{\text{op}}.$$

Proof. (1) This is statement [14, Lemma 6.1]. (2) The proof of the first equivalence goes *mutatis mutandis* as the one of [16, Prop. 3.15]. The second equivalence is straightforward. \square

2.6 Tensor product of bimodule categories over Hopf algebras

Let A, B be finite-dimensional Hopf algebras. A $(\text{Rep}(B), \text{Rep}(A))$ -bimodule category is the same as a left $\text{Rep}(B \otimes_{\mathbb{k}} A^{\text{cop}})$ -module category. By [1, Theorem 3.3] we know that any exact indecomposable $(\text{Rep}(B), \text{Rep}(A))$ -bimodule category is equivalent to the category ${}_S \mathcal{M}$ of finite-dimensional left S -modules, where S is a finite-dimensional right $B \otimes_{\mathbb{k}} A^{\text{cop}}$ -simple left $B \otimes_{\mathbb{k}} A^{\text{cop}}$ -comodule algebra.

Remark 2.6 The identity object in $\text{BrPic}(\text{Rep}(A))$ is the class of the $\text{Rep}(A)$ -bimodule category $\text{diag}(A) \mathcal{M}$, where $\text{diag}(A) = A$ as algebras, and the left $A \otimes_{\mathbb{k}} A^{\text{cop}}$ -comodule structure is given by:

$$\lambda : \text{diag}(A) \rightarrow A \otimes_{\mathbb{k}} A^{\text{cop}} \otimes_{\mathbb{k}} \text{diag}(A), \quad \lambda(a) = a_{(1)} \otimes a_{(3)} \otimes a_{(2)}, \quad a \in A.$$

We proceed to determine the tensor product over $\text{Rep}(B)$ of a $(\text{Rep}(A), \text{Rep}(B))$ -bimodule category and a $(\text{Rep}(B), \text{Rep}(A))$ -bimodule category, both exact indecomposable. Throughout, for such a product we shall shortly say *tensor product of bimodule categories over a Hopf algebra*.

Define $\pi_A : A \otimes B \rightarrow A$, $\pi_B : A \otimes B \rightarrow B$ the algebra maps

$$\pi_A(x \otimes y) = \epsilon(y)x, \quad \pi_B(x \otimes y) = \epsilon(x)y,$$

for all $x \in A, y \in B$.

Let K be a right $B \otimes A^{\text{cop}}$ -simple left $B \otimes A^{\text{cop}}$ -comodule algebra and L a right $A \otimes B^{\text{cop}}$ -simple left $A \otimes B^{\text{cop}}$ -comodule algebra. Thus the category ${}_K \mathcal{M}$ is a $(\text{Rep}(B), \text{Rep}(A))$ -bimodule category and ${}_L \mathcal{M}$ is a $(\text{Rep}(A), \text{Rep}(B))$ -bimodule category.

Recall that \overline{L} is the left $B \otimes A^{\text{cop}}$ -comodule algebra with opposite algebra structure L^{op} and left $B \otimes A^{\text{cop}}$ -comodule structure:

$$\overline{\lambda} : L \rightarrow B \otimes_{\mathbb{k}} A^{\text{cop}} \otimes_{\mathbb{k}} L, \quad l \mapsto (\mathcal{S}_B^{-1} \otimes \mathcal{S}_A)(l_{(-1)}) \otimes l_{(0)}, \quad (8)$$

for all $l \in L$.

We denote by ${}_K^B \mathcal{M}_{\overline{L}}$ the category of (K, \overline{L}) -bimodules and left B -comodules such that the comodule structure is a bimodule morphism. See [19, Section 3]. It has a structure of $\text{Rep}(A)$ -bimodule category.

Also L is a right B -comodule and K is a left B -comodule with comodule maps given by

$$l \mapsto l_{(0)} \otimes \pi_B(l_{(-1)}), \quad k \mapsto \pi_B(k_{(-1)}) \otimes k_{(0)}, \quad (9)$$

for all $l \in L, k \in K$. Using this structure we can form the cotensor product $L \square_B K$. Define

$$\lambda(l \otimes k) = \pi_A(l_{(-1)}) \otimes \pi_A(k_{(-1)}) \otimes l_{(0)} \otimes k_{(0)}, \quad (10)$$

for all $l \otimes k \in L \square_B K$. Then $L \square_B K$ is a left $A \otimes_{\mathbb{k}} A^{\text{cop}}$ -comodule algebra. See [19, Lemma 3.6].

Recall that in [19] we have defined a structure of $\text{Rep}(A)$ -bimodule category on ${}^B_K\mathcal{M}_{\overline{L}}$. Similarly, we can define a $\text{Rep}(A)$ -bimodule category structure on ${}_L\mathcal{M}_{\overline{K}}^B$.

Theorem 2.7 (a) *There is a $\text{Rep}(A)$ -bimodule equivalence:*

$${}_L\mathcal{M} \boxtimes_{\text{Rep}(B)} {}_K\mathcal{M} \simeq {}^B_K\mathcal{M}_{\overline{L}}.$$

(b) *The functor I from Lemma 2.3 is an equivalence of $\text{Rep}(A)$ -bimodule categories.*

(c) *If L is a Hopf-Galois extension, as a right B -comodule algebra, then there is an equivalence of $\text{Rep}(A)$ -bimodule categories*

$${}_L\Box_B {}_K\mathcal{M} \simeq {}_L\mathcal{M} \boxtimes_{\text{Rep}(B)} {}_K\mathcal{M}.$$

Proof. Item (a) was proven in [19]. Item (b) is straightforward and (c) follows from Theorem 2.2 and items (a) and (b).

3 Bicategories and tensor categories

For a review on basic notions on bicategories we refer to [3, 6]. When the associativity constraint of a bicategory is the identity, the bicategory is said to be a 2-category. For completeness we add that a 2-equivalence between 2-categories \mathcal{B} and \mathcal{B}' is a pseudo-functor $(\Theta, \theta) : \mathcal{B} \rightarrow \mathcal{B}'$ such that there is another pseudo-functor $(\Pi, \pi) : \mathcal{B}' \rightarrow \mathcal{B}$ and two pseudo-natural isomorphisms $\sigma : (\Pi, \pi) \circ (\Theta, \theta) \rightarrow \text{Id}_{\mathcal{B}}$ and $\tau : (\Theta, \theta) \circ (\Pi, \pi) \rightarrow \text{Id}_{\mathcal{B}'}$.

It is well-known that any monoidal category \mathcal{C} gives rise to a bicategory with only one object. We shall denote by $\underline{\mathcal{C}}$ this bicategory. If \mathcal{C}, \mathcal{D} are strict monoidal categories, a pseudo-functor $(F, \xi) : \underline{\mathcal{C}} \rightarrow \underline{\mathcal{D}}$ is nothing but a monoidal functor between \mathcal{C} and \mathcal{D} . If $(F, \xi), (G, \zeta) : \mathcal{C} \rightarrow \mathcal{D}$ are monoidal functors between two strict monoidal categories, a pseudo-natural transformation between them is a pair $(\eta, \eta_0) : (F, \xi) \rightarrow (G, \zeta)$ where $\eta_0 \in \mathcal{D}$ is an object and for any $X \in \mathcal{C}$ natural transformations

$$\eta_X : F(X) \otimes \eta_0 \rightarrow \eta_0 \otimes G(X),$$

such that

$$(\text{id}_{\eta_0} \otimes \zeta_{X,Y})\eta_{X \otimes Y} = (\eta_X \otimes \text{id}_{G(Y)})(\text{id}_{F(X)} \otimes \eta_Y)(\xi_{X,Y} \otimes \text{id}_{\eta_0}) \quad (11)$$

Set $\tilde{\xi}_X$ for the natural isomorphism $\xi_{1,X} : F(X) \rightarrow F(X)$ (and similarly for $\zeta_{1,X}$). Then $\eta_{1_{\mathcal{C}}}$ is a morphism $\eta_1 : \eta_0 \rightarrow \eta_0$ in \mathcal{D} satisfying

$$(\text{id}_{\eta_0} \otimes \tilde{\xi}_X)\eta_X = (\eta_1 \otimes \text{id}_{G(X)})\eta_X(\tilde{\xi}_X \otimes \text{id}_{\eta_0}).$$

for every $X \in \mathcal{C}$. Given two pseudo-natural transformations $(\eta, \eta_0) : (F, \xi) \rightarrow (G, \zeta)$ and $(\sigma, \sigma_0) : (G, \zeta) \rightarrow (H, \chi)$ their composition is given by $((\text{id}_{\eta_0} \otimes \sigma)(\eta \otimes \text{id}_{\sigma_0}), \eta_0 \otimes \sigma_0) : (F, \xi) \rightarrow (H, \chi)$. A pair (η, η_0) is a pseudo-natural isomorphism if there exists a pseudo-natural transformation (σ, σ_0) such that $(\eta, \eta_0)(\sigma, \sigma_0) = (\text{id}_F, 1_{\mathcal{D}})$ and $(\sigma, \sigma_0)(\eta, \eta_0) = (\text{id}_G, 1_{\mathcal{D}})$. Consequently, the object η_0 is invertible in \mathcal{D} , that is, there exists an object $\overline{\eta_0} \in \mathcal{D}$ such that $\eta_0 \otimes \overline{\eta_0} = 1_{\mathcal{D}} = \overline{\eta_0} \otimes \eta_0$. Any natural monoidal transformation $\mu : (F, \xi) \rightarrow (G, \zeta)$ gives rise to a pseudo-natural transformation $(\mu, \mathbf{1})$.

Lemma 3.1 *Let (\mathcal{D}, c) be a strict braided monoidal category. Then any pseudo-natural isomorphism $(\eta, \eta_0) : (F, \xi) \rightarrow (G, \zeta)$ between two monoidal functors as above produces a natural monoidal isomorphism.*

Proof. Given a pseudo-natural isomorphism $(\eta, \eta_0) : (F, \xi) \rightarrow (G, \zeta)$ define $\mu : (F, \xi) \rightarrow (G, \zeta)$ as the composition

$$F(X) \xrightarrow{\eta_0} F(X) \otimes \eta_0 \otimes \overline{\eta_0} \xrightarrow{\eta_X \otimes \text{id}} \eta_0 \otimes G(X) \otimes \overline{\eta_0} \xrightarrow{c_{\eta_0, G(X)} \otimes \text{id}} G(X) \otimes \eta_0 \otimes \overline{\eta_0} \xrightarrow{\eta_0} G(X)$$

for any $X \in \mathcal{C}$. Here $\overline{\eta_0}$ is the inverse object of η_0 . Then μ is clearly a natural transformation. We prove that it is monoidal:

$$\begin{aligned} \zeta_{X,Y} \mu(X \otimes Y) &= ((\zeta_{X,Y} \otimes \text{id}_{\eta_0})(c_{\eta_0, G(X \otimes Y)} \circ \eta_{X \otimes Y}) \otimes \text{id}_{\overline{\eta_0}}) (\text{id}_{F(X \otimes Y)} \otimes \eta_0 \otimes \overline{\eta_0}) \\ &= ((\text{id}_{G(X)} \otimes c_{\eta_0, G(Y)})(c_{\eta_0, G(X)} \otimes \text{id}_{G(Y)})(\text{id}_{\eta_0} \otimes \zeta_{X,Y}) \eta_{X \otimes Y} \otimes \text{id}_{\overline{\eta_0}}) (\text{id}_{F(X \otimes Y)} \otimes \eta_0 \otimes \overline{\eta_0}) \\ &= ((\text{id}_{G(X)} \otimes \text{id}_{\eta_0 \otimes \overline{\eta_0}} \otimes c_{\eta_0, G(Y)})(c_{\eta_0, G(X)} \otimes \text{id}_{\overline{\eta_0} \otimes \eta_0} \otimes \text{id}_{G(Y)})(\eta_X \otimes \text{id}_{\overline{\eta_0} \otimes \eta_0} \otimes \text{id}_{G(Y)}) \\ &\quad (\text{id}_{F(X)} \otimes \eta_0 \otimes \overline{\eta_0} \otimes \eta_Y)(\xi_{X,Y} \otimes \text{id}_{\eta_0}) \otimes \text{id}_{\overline{\eta_0}}) (\text{id}_{F(X \otimes Y)} \otimes \eta_0 \otimes \overline{\eta_0}) \\ &= (((c_{\eta_0, G(X)} \otimes \eta_X) \otimes \text{id}_{\overline{\eta_0}})(\text{id}_{F(X)} \otimes \eta_0 \otimes \overline{\eta_0}) \otimes ((c_{\eta_0, G(Y)} \circ \eta_Y) \otimes \text{id}_{\overline{\eta_0}})(\text{id}_{F(Y)} \otimes \eta_0 \otimes \overline{\eta_0})) \xi_{X,Y} \\ &= (\mu(X) \otimes \mu(Y)) \xi_{X,Y}. \end{aligned}$$

The second equality holds by naturality of the braiding and the third one is due to (11) and because $\eta_0 \otimes \overline{\eta_0} = \mathbf{1}$. \square

Remark 3.2 In Lemma 3.1 the category \mathcal{D} need not necessarily be braided. Observe that it suffices that the object η_0 is equipped with a lift to the Drinfel'd center $\mathcal{Z}(\mathcal{D})$ of \mathcal{D} . That is, that there exists a family of natural isomorphisms $c_{\eta_0, X} : \eta_0 \otimes X \rightarrow X \otimes \eta_0$ satisfying $c_{\eta_0, X \otimes Y} = (X \otimes c_{\eta_0, Y})(c_{\eta_0, X} \otimes Y)$, for all $X, Y \in \mathcal{D}$.

Let \mathcal{C} be a finite tensor category. We denote by $\mathcal{C}\text{-Mod}$ the 2-category (of \mathcal{C} -module categories) whose 0-cells are \mathcal{C} -module categories, 1-cells are \mathcal{C} -module functors between them and 2-cells are natural transformations between such functors (i.e. for two 0-cells \mathcal{M}, \mathcal{N} there is a category $\text{Func}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ whose objects and morphisms present the 1- and 2-cells of $\mathcal{C}\text{-Mod}$). A \mathcal{C} -module functor $\mathcal{F} : \mathcal{M} \rightarrow \mathcal{N}$ is equipped with a natural isomorphism $c_{X, M} : \mathcal{F}(X \overline{\otimes} M) \rightarrow X \overline{\otimes} \mathcal{F}(M)$ for $X \in \mathcal{C}, M \in \mathcal{M}$. Then a natural transformation between two such functors (\mathcal{F}, c) and (\mathcal{G}, d) is $\alpha : \mathcal{F} \rightarrow \mathcal{G}$ such that $(X \overline{\otimes} \alpha(M))c_{X, M} = d_{X, M} \alpha(X \overline{\otimes} M)$.

Remark 3.3 For a left \mathcal{C} -module category \mathcal{M} the action functor $\overline{\otimes}$ is biexact. Therefore, for a $\mathcal{C}\text{-}\mathcal{D}$ -bimodule category \mathcal{M} and $\mathcal{N} \in \mathcal{D}\text{-Mod}$ we have canonical isomorphisms $X \overline{\otimes} (M \boxtimes_{\mathcal{D}} N) \cong (X \overline{\otimes} M) \boxtimes_{\mathcal{D}} N$ for all $X \in \mathcal{C}, M \in \mathcal{M}, N \in \mathcal{N}$.

The following result seems to be well-known. We write the proof for the reader's sake.

Theorem 3.4 *Two finite tensor categories \mathcal{C} and \mathcal{D} are Morita equivalent if and only if there is a 2-equivalence $(\mathcal{H}, \mathfrak{h}) : \mathcal{D}\text{-Mod} \rightarrow \mathcal{C}\text{-Mod}$.*

Proof. Let $(\mathcal{H}, \mathfrak{h}) : \mathcal{D}\text{-Mod} \rightarrow \mathcal{C}\text{-Mod}$ be a 2-equivalence. For two 0-cells \mathcal{N}, \mathcal{L} we have an equivalence functor $\mathcal{H}_{\mathcal{N}, \mathcal{L}} : \text{Fun}_{\mathcal{D}}(\mathcal{N}, \mathcal{L}) \rightarrow \text{Func}(\mathcal{H}(\mathcal{N}), \mathcal{H}(\mathcal{L}))$ equipped with a monoidal structure \mathfrak{h} for the composition of 1-cells. Let $\mathcal{N}, \mathcal{L}, \mathcal{P} \in \mathcal{D}\text{-Mod}$ and let $\mathcal{F} : \mathcal{N} \rightarrow \mathcal{L}$ and $\mathcal{G} : \mathcal{L} \rightarrow \mathcal{P}$ be two 1-cells. There is a natural isomorphism

$$\mathfrak{h}^{\mathcal{N}, \mathcal{L}, \mathcal{P}} : \mathcal{H}_{\mathcal{N}, \mathcal{P}}(\mathcal{F} \circ \mathcal{G}) \rightarrow \mathcal{H}_{\mathcal{N}, \mathcal{L}}(\mathcal{F}) \circ \mathcal{H}_{\mathcal{L}, \mathcal{P}}(\mathcal{G}) \quad (12)$$

(we usually write these in the reverted order of \mathcal{F} and \mathcal{G}). Then $\text{End}_{\mathcal{C}}(\mathcal{H}(\mathcal{D})) = \text{Func}(\mathcal{H}(\mathcal{D}), \mathcal{H}(\mathcal{D})) \simeq \text{Fun}_{\mathcal{D}}(\mathcal{D}, \mathcal{D}) \simeq \mathcal{D}$ as monoidal categories, thus \mathcal{C} and \mathcal{D} are Morita equivalent.

If \mathcal{C} and \mathcal{D} are Morita equivalent, by Lemma 2.4, there is an invertible $(\mathcal{C}, \mathcal{D})$ -bimodule category \mathcal{M} which gives rise to the desired 2-equivalence functor $(\mathcal{H}, \mathfrak{h}) : \mathcal{D}\text{-Mod} \rightarrow \mathcal{C}\text{-Mod}$. On 0-cells define $\mathcal{H} = \mathcal{M} \boxtimes_{\mathcal{D}} - : \mathcal{D}\text{-Mod} \rightarrow \mathcal{C}\text{-Mod}$, that is $\mathcal{H}(\mathcal{N}) = \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$ for $\mathcal{N} \in \mathcal{D}\text{-Mod}$. For two 0-cells \mathcal{N}, \mathcal{L} define the functor $\mathcal{H}_{\mathcal{N}, \mathcal{L}} : \text{Fun}_{\mathcal{D}}(\mathcal{N}, \mathcal{L}) \rightarrow \text{Func}(\mathcal{H}(\mathcal{N}), \mathcal{H}(\mathcal{L}))$ by $\mathcal{H}_{\mathcal{N}, \mathcal{L}} = \mathcal{M} \boxtimes_{\mathcal{D}} -$, i.e. for a \mathcal{D} -module functor $\mathcal{F} : \mathcal{N} \rightarrow \mathcal{L}$ we have $\mathcal{H}_{\mathcal{N}, \mathcal{L}}(\mathcal{F}) = \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{F} : \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \rightarrow \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{L}$. For objects $M \in \mathcal{M}, N \in \mathcal{N}$ it is $(\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{F})(M \boxtimes_{\mathcal{D}} N) = M \boxtimes_{\mathcal{D}} \mathcal{F}(N)$. If \mathcal{F} is a left \mathcal{D} -module functor then $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{F}$ is a left \mathcal{C} -module functor with the canonical isomorphism: $\tilde{c}_{Y, M \boxtimes_{\mathcal{D}} N} : (\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{F})(Y \otimes (M \boxtimes_{\mathcal{D}} N)) \rightarrow Y \otimes (\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{F})(M \boxtimes_{\mathcal{D}} N) = Y \otimes (M \boxtimes_{\mathcal{D}} \mathcal{F}(N))$ where $Y \in \mathcal{C}$ and $M \in \mathcal{M}, N \in \mathcal{N}$. Due to Remark 3.3 the source object of $\tilde{c}_{Y, M \boxtimes_{\mathcal{D}} N}$ is isomorphic to $(Y \otimes M) \boxtimes_{\mathcal{D}} \mathcal{F}(N)$ which is clearly isomorphic to the target object. For two 1-cells $\mathcal{F}, \mathcal{G} : \mathcal{N} \rightarrow \mathcal{L}$ and a 2-cell $\alpha : \mathcal{F} \rightarrow \mathcal{G}$ we define $\mathcal{H}_{\mathcal{N}, \mathcal{L}}(\alpha) = \mathcal{M} \boxtimes_{\mathcal{D}} \alpha : \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{F} \rightarrow \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{G}$ by $(\mathcal{M} \boxtimes_{\mathcal{D}} \alpha)(M \boxtimes_{\mathcal{D}} N) = M \boxtimes_{\mathcal{D}} \alpha(N) : M \boxtimes_{\mathcal{D}} \mathcal{F}(N) \rightarrow M \boxtimes_{\mathcal{D}} \mathcal{G}(N)$. The natural transformation $\mathcal{M} \boxtimes_{\mathcal{D}} \alpha$ fulfills the compatibility condition $(Y \otimes (\mathcal{M} \boxtimes_{\mathcal{D}} \alpha)(M \boxtimes_{\mathcal{D}} N)) \tilde{c}_{Y, M \boxtimes_{\mathcal{D}} N} = \tilde{d}_{Y, M \boxtimes_{\mathcal{D}} N}(\mathcal{M} \boxtimes_{\mathcal{D}} \alpha)(Y \otimes (M \boxtimes_{\mathcal{D}} N))$ by Remark 3.3 and since $\tilde{c}_{Y, M \boxtimes_{\mathcal{D}} N}$ and $\tilde{d}_{Y, M \boxtimes_{\mathcal{D}} N}$ are canonical isomorphisms.

Observe that a monoidal structure \mathfrak{h} for the composition of 1-cells (12) in this case is an isomorphism from $\mathcal{M} \boxtimes_{\mathcal{D}} (\mathcal{F} \circ \mathcal{G})$ to $(\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{F}) \circ (\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{G})$. Though, these two functors are equal, so we take $\mathfrak{h}^{\mathcal{N}, \mathcal{L}, \mathcal{P}}$ to be the identity for all $\mathcal{N}, \mathcal{L}, \mathcal{P} \in \mathcal{D}\text{-Mod}$.

Since \mathcal{M} is invertible, the pseudo-functor $(\mathcal{H}, \mathfrak{h})$ is a 2-equivalence. \square

4 Bi-Galois objects and invertible bimodule categories

Let H, L be finite-dimensional Hopf algebras. An (H, L) -biGalois object, introduced by Schauenburg in [22], is an algebra A that is a left H -Galois extension and a right L -Galois extension of \mathbb{k} such that the two comodule structures make it an (H, L) -bicomodule. Two biGalois objects are isomorphic if there exists a bijective bicomodule morphism that is also an algebra map. Denote by $\text{BiGal}(H)$ the set of isomorphism classes of (H, H) -biGalois extensions. It is a group with product given by \square_H .

If A is an (H, L) -biGalois object then the functor $\mathcal{F}_A : \text{Comod}(L) \rightarrow \text{Comod}(H)$, $\mathcal{F}_A(X) = A \square_L X$, $X \in \text{Comod}(L)$, is a tensor equivalence functor [24]. The tensor structure on \mathcal{F}_A is as follows. If $X, Y \in \text{Comod}(L)$ then

$$\xi_{X, Y}^A : (A \square_L X) \otimes_{\mathbb{k}} (A \square_L Y) \rightarrow A \square_L (X \otimes_{\mathbb{k}} Y), \quad \xi_{X, Y}^A(a_i \otimes x_i \otimes b_j \otimes y_j) = a_i b_j \otimes x_i \otimes y_j \quad (13)$$

for any $a_i \otimes x_i \in A \square_L X$, $b_j \otimes y_j \in A \square_L Y$. If A, B are (H, L) -biGalois objects then there is a natural monoidal isomorphism between the tensor functors $\mathcal{F}_A, \mathcal{F}_B$ if and only if $A \simeq B$ as biGalois objects, [22, Corollary 5.7].

Lemma 4.1 *Let A be an (H, H) -biGalois object.*

1. *The category ${}_A\mathcal{M}$ is an invertible $\text{Rep}(H)$ -bimodule category.*
2. *$\text{Comod}(H)^{\mathcal{F}^A}$ is an invertible $\text{Comod}(H)$ -bimodule category.*

Proof. (1) is a consequence of Theorem 2.7 (c) and (2) is a particular case of Lemma 2.5. \square

Bimodule categories in Lemma 4.1 are related via the isomorphism presented in (7). Let us explain this assertion in detail. The category of finite-dimensional vector spaces $\text{vect}_{\mathbb{k}}$ is an invertible $(\text{Comod}(H), \text{Rep}(H^{\text{op}}))$ -bimodule category. Let us denote

$$\Phi : \text{BrPic}(\text{Comod}(H)) \rightarrow \text{BrPic}(\text{Rep}(H^{\text{op}})) \quad (14)$$

the isomorphism described in (7) using $\mathcal{M} = \text{vect}_{\mathbb{k}}$.

Proposition 4.2 *Let A be an (H, H) -biGalois object, then $\Phi([\text{Comod}(H)^{\mathcal{F}^A}]) = [\mathcal{M}_A]$.*

Proof. Let us denote $\mathcal{C} = \text{Comod}(H)$. By definition we get

$$\begin{aligned} \Phi([\mathcal{C}^{\mathcal{F}^A}]) &= [\text{vect}_{\mathbb{k}}^{\text{op}} \boxtimes_{\mathcal{C}} \mathcal{C}^{\mathcal{F}^A} \boxtimes_{\mathcal{C}} \text{vect}_{\mathbb{k}}] \\ &= [(\text{vect}_{\mathbb{k}}^{\text{op}})^{\mathcal{F}^A} \boxtimes_{\mathcal{C}} \text{vect}_{\mathbb{k}}] \\ &= [({}^{\mathcal{F}^A}\text{vect}_{\mathbb{k}})^{\text{op}} \boxtimes_{\mathcal{C}} \text{vect}_{\mathbb{k}}] \\ &= [\text{Hom}_{\mathcal{C}}({}^{\mathcal{F}^A}\text{vect}_{\mathbb{k}}, \text{vect}_{\mathbb{k}})]. \end{aligned}$$

The second and third equality follow from Lemma 2.5 (2). The last equality is [16, Thm. 3.20]. It remains to prove that there is an equivalence of bimodule categories

$$\mathcal{M}_A \simeq \text{Hom}_{\mathcal{C}}({}^{\mathcal{F}^A}\text{vect}_{\mathbb{k}}, \text{vect}_{\mathbb{k}}).$$

We shall only sketch the proof. Given an object $(U, \mu) \in \mathcal{M}_A$, $\mu : U \otimes_{\mathbb{k}} A \rightarrow U$, define the functor $(G, c) \in \text{Hom}_{\mathcal{C}}({}^{\mathcal{F}^A}\text{vect}_{\mathbb{k}}, \text{vect}_{\mathbb{k}})$ as follows. For any $M \in \text{vect}_{\mathbb{k}}$ set $G(M) = U \otimes_{\mathbb{k}} M$, and $c_{X, M} : U \otimes_{\mathbb{k}} (A \square_H X) \otimes_{\mathbb{k}} M \rightarrow X \otimes_{\mathbb{k}} U \otimes_{\mathbb{k}} M$ is

$$c_{X, M}(u \otimes a_i \otimes x_i \otimes m) = x_i \otimes u \cdot a_i \otimes m,$$

for all $u \in U$, $m \in M$, $\sum a_i \otimes x_i \in A \square_H X$. Conversely, given a module functor $(G, c) \in \text{Hom}_{\mathcal{C}}({}^{\mathcal{F}^A}\text{vect}_{\mathbb{k}}, \text{vect}_{\mathbb{k}})$, since it is exact, there exists an object $U \in \text{vect}_{\mathbb{k}}$ such that $G(M) = U \otimes_{\mathbb{k}} M$ for any $M \in \text{vect}_{\mathbb{k}}$. The object U has a right A -module structure $\mu : U \otimes_{\mathbb{k}} A \rightarrow U$ defined by

$$\mu = (\epsilon \otimes \text{id}_U) c_{H, \mathbb{k}}(\text{id}_U \otimes \rho).$$

Here $\rho : A \rightarrow A \otimes_{\mathbb{k}} H$ is the right H -comodule structure. Both constructions are well-defined and inverse of each other. \square

If (A, λ) is a left H -comodule algebra and $g \in G(H)$ is a group-like element we can define a new comodule algebra A^g on the same underlying algebra A with the coaction given by $\lambda^g : A^g \rightarrow H \otimes_{\mathbb{k}} A^g$:

$$\lambda^g(a) = g^{-1} a_{(-1)} g \otimes a_{(0)} \quad (15)$$

for all $a \in A$. If A is an (H, H) -biGalois object let A^g denote the above left comodule algebra whose right comodule structure remains unchanged.

Lemma 4.3 A^g is an (H, H) -biGalois object. □

Definition 4.4 If $A, B \in \text{BiGal}(H)$ we shall say that A is equivalent to B , and denote it by $A \sim B$ if there exists an element $g \in G(H)$ such that $A^g \simeq B$ as biGalois objects.

Theorem 4.5 Let $A, B \in \text{BiGal}(H)$. The following statements are equivalent.

1. $A \sim B$;
2. there exists an equivalence $\text{Comod}(H)^{\mathcal{F}_A} \simeq \text{Comod}(H)^{\mathcal{F}_B}$ of $\text{Comod}(H)$ -bimodule categories;
3. there exists an equivalence ${}_A\mathcal{M} \simeq {}_B\mathcal{M}$ of $\text{Rep}(H)$ -bimodule categories;
4. there exists a pseudo-natural isomorphism $(\eta, \eta_0) : \mathcal{F}_A \rightarrow \mathcal{F}_B$.

Proof. The equivalence between (2) and (4) is given in [14, Lemma 6.1]. The equivalence between (2) and (3) follows from Proposition 4.2. Let us prove that (1) is equivalent to (4). Assume that there is a group-like element $g \in G(H)$ and a bicomodule algebra isomorphism $f : A^g \rightarrow B$. Define $\eta_0 = \mathbb{k}$ with left H -comodule action $\eta_0 \rightarrow H \otimes_{\mathbb{k}} \eta_0$, $1 \rightarrow g \otimes 1$, and for any $X \in \text{Comod}(H)$

$$\eta_X : \mathcal{F}_A(X) \otimes_{\mathbb{k}} \eta_0 \rightarrow \eta_0 \otimes_{\mathbb{k}} \mathcal{F}_B(X), \quad \eta_X(a \otimes x \otimes 1) = 1 \otimes f(a) \otimes x,$$

for all $a \otimes x \in \mathcal{F}_A(X)$. Since f is a right H -comodule morphism the map η_X is well-defined. Let $X, Y \in \text{Comod}(H)$, $a \otimes x \in \mathcal{F}_A(X)$, $b \otimes y \in \mathcal{F}_A(Y)$, then

$$\begin{aligned} (\text{id}_{\eta_0} \otimes (\xi_{X,Y}^B)^{-1})(\eta_X \otimes \text{id})(\text{id} \otimes \eta_Y)(a \otimes x \otimes b \otimes y \otimes 1) &= 1 \otimes f(a)f(b) \otimes x \otimes y \\ &= 1 \otimes f(ab) \otimes x \otimes y \\ &= \eta_{X \otimes Y}((\xi_{X,Y}^A)^{-1} \otimes \text{id}_{\eta_0})(a \otimes x \otimes b \otimes y \otimes 1). \end{aligned}$$

Thus (11) is fulfilled and (η, η_0) is a pseudo-natural transformation.

Now, let us assume that there exists a pseudo-natural isomorphism $(\eta, \eta_0) : \mathcal{F}_A \rightarrow \mathcal{F}_B$. Since $\eta_0 \in \text{Comod}(H)$ is an invertible object it is one-dimensional. Hence, there exists a group-like element $g \in G(H)$ such that the coaction $\eta_0 \rightarrow H \otimes_{\mathbb{k}} \eta_0$ is given by $1 \mapsto g \otimes 1$. Define $f : A \rightarrow B$ as the composition

$$A \xrightarrow{\iota} A \otimes_{\mathbb{k}} \eta_0 \xrightarrow{\rho \otimes \text{id}} A \square_H H \otimes_{\mathbb{k}} \eta_0 \xrightarrow{\eta_H} \eta_0 \otimes_{\mathbb{k}} B \square_H H \xrightarrow{\text{id} \otimes \epsilon} \eta_0 \otimes_{\mathbb{k}} B \xrightarrow{\pi} B.$$

We must show that f is an algebra map and an H -bicomodule homomorphism.

Claim 4.6 $f : A \rightarrow B$ is an algebra map.

Proof of Claim. It is enough to prove that η_H is an algebra map. Observe that $A \square_H H$ is a subalgebra of $A \otimes_{\mathbb{k}} H$ and the algebra structure on $A \square_H H \otimes_{\mathbb{k}} \eta_0$ is that of the tensor product algebra. We shall denote

$$m_1 : \mathcal{F}_A(H) \otimes_{\mathbb{k}} \eta_0 \otimes_{\mathbb{k}} \mathcal{F}_A(H) \otimes_{\mathbb{k}} \eta_0 \rightarrow \mathcal{F}_A(H) \otimes_{\mathbb{k}} \eta_0,$$

$$m_2 : \eta_0 \otimes_{\mathbb{k}} \mathcal{F}_B(H) \otimes_{\mathbb{k}} \eta_0 \otimes_{\mathbb{k}} \mathcal{F}_B(H) \rightarrow \eta_0 \otimes_{\mathbb{k}} \mathcal{F}_B(H),$$

the algebra structures. Define the isomorphisms

$$\begin{aligned}\gamma_0 : \eta_0 \otimes_{\mathbb{k}} \mathcal{F}_B(H) \otimes_{\mathbb{k}} \mathcal{F}_B(H) &\rightarrow \eta_0 \otimes_{\mathbb{k}} \mathcal{F}_B(H) \otimes_{\mathbb{k}} \eta_0 \otimes_{\mathbb{k}} \mathcal{F}_B(H), & \gamma_0(1 \otimes a \otimes b) &= 1 \otimes a \otimes 1 \otimes b, \\ \gamma_1 : \mathcal{F}_A(H) \otimes_{\mathbb{k}} \mathcal{F}_A(H) \otimes_{\mathbb{k}} \eta_0 &\rightarrow \mathcal{F}_A(H) \otimes_{\mathbb{k}} \eta_0 \otimes_{\mathbb{k}} \mathcal{F}_A(H) \otimes_{\mathbb{k}} \eta_0, & \gamma_1(x \otimes y \otimes 1) &= x \otimes 1 \otimes y \otimes 1, \\ \gamma_2 : \mathcal{F}_A(H) \otimes_{\mathbb{k}} \eta_0 \otimes \mathcal{F}_B(H) &\rightarrow \mathcal{F}_A(H) \otimes_{\mathbb{k}} \eta_0 \otimes_{\mathbb{k}} \eta_0 \otimes_{\mathbb{k}} \mathcal{F}_B(H), & \gamma_2(x \otimes 1 \otimes b) &= x \otimes 1 \otimes 1 \otimes b,\end{aligned}$$

for all $a, b \in \mathcal{F}_B(H)$, $x, y \in \mathcal{F}_A(H)$. It is not difficult to prove that

$$(\text{id}_{\mathcal{F}_A(H)} \otimes \text{id}_{\eta_0} \otimes \eta_H) \gamma_1 = \gamma_2 (\text{id}_{\mathcal{F}_A(H)} \otimes \eta_H), \quad (16)$$

$$(\eta_H \otimes \text{id}_{\eta_0} \otimes \text{id}_{\mathcal{F}_B(H)}) \gamma_2 = \gamma_0 (\eta_H \otimes \text{id}_{\mathcal{F}_B(H)}). \quad (17)$$

Let us denote $m : H \otimes_{\mathbb{k}} H \rightarrow H$ the product. Since m is a morphism in $\text{Comod}(H)$, the naturality of η implies that $(\text{id}_{\eta_0} \otimes \mathcal{F}_B(m)) \eta_{H \otimes H} = \eta_H(\mathcal{F}_A(m) \otimes \text{id}_{\eta_0})$. The following equalities are readily verified:

$$\mathcal{F}_A(m) \otimes \text{id}_{\eta_0} = m_1 \gamma_1(\xi_{H,H}^A \otimes \text{id}_{\eta_0}), \quad \text{id}_{\eta_0} \otimes \mathcal{F}_B(m) = m_2 \gamma_0(\text{id}_{\eta_0} \otimes \xi_{H,H}^B). \quad (18)$$

Since η is pseudo-natural, then

$$(\eta_H \otimes \text{id}_{\mathcal{F}_B(H)})(\text{id}_{\mathcal{F}_A(H)} \otimes \eta_H)(\xi_{H,H}^A \otimes \text{id}_{\eta_0}) = (\text{id}_{\eta_0} \otimes \xi_{H,H}^B) \eta_{H \otimes H}. \quad (19)$$

Let us denote the isomorphism $\phi = \gamma_1(\xi_{H,H}^A \otimes \text{id}_{\eta_0})$. We have that

$$\begin{aligned}(\eta_H \otimes \eta_H) \phi &= (\eta_H \otimes \text{id}_{\eta_0 \otimes \mathcal{F}_B(H)})(\text{id}_{\mathcal{F}_A(H) \otimes \eta_0} \otimes \eta_H) \gamma_1(\xi_{H,H}^A \otimes \text{id}_{\eta_0}) \\ &= (\eta_H \otimes \text{id}_{\eta_0 \otimes \mathcal{F}_B(H)})(\gamma_2(\text{id}_{\mathcal{F}_A(H)} \otimes \eta_H))(\xi_{H,H}^A \otimes \text{id}_{\eta_0}) \\ &= \gamma_0(\eta_H \otimes \text{id}_{\mathcal{F}_B(H)})(\text{id}_{\mathcal{F}_A(H)} \otimes \eta_H)(\xi_{H,H}^A \otimes \text{id}_{\eta_0}) \\ &= \gamma_0(\text{id}_{\eta_0} \otimes \xi_{H,H}^B) \eta_{H \otimes H}.\end{aligned}$$

The second equality follows from (16), the third equality by (17) and the last equality follows from (19). Now, we have that

$$\begin{aligned}\eta_H m_1 \phi &= \eta_H(\mathcal{F}_A(m) \otimes \text{id}_{\eta_0}) = (\text{id}_{\eta_0} \otimes \mathcal{F}_B(m)) \eta_{H \otimes H} \\ &= m_2 \gamma_0(\text{id}_{\eta_0} \otimes \xi_{H,H}^B) \eta_{H \otimes H} = m_2 (\eta_H \otimes \eta_H) \phi.\end{aligned}$$

The first and third equalities follow from (18). Hence $\eta_H m_1 = m_2 (\eta_H \otimes \eta_H)$ and η_H is an algebra map. That $\eta_H(1 \otimes 1 \otimes 1) = 1 \otimes 1 \otimes 1$ follows from the naturality of η (since $A \square_H \mathbb{k} = \mathbb{k}$, it is $\eta_{\mathbb{k}} = \text{id}_{\mathbb{k}}$). \square

For any vector space V we shall denote by V^t the same vector space V with trivial left H -coaction: $\lambda^t : V^t \rightarrow H \otimes_{\mathbb{k}} V^t$, $\lambda^t(v) = 1 \otimes v$ for all $v \in V$. Then $A \square_H V^t = V^t$ and η is additive, because for any vector space V we have that $\eta_{V^t} = \text{id}_V$.

Claim 4.7 $f : A \rightarrow B$ is a right H -comodule map.

Proof of Claim. The spaces $A \square_H H \otimes_{\mathbb{k}} \eta_0$ and $\eta_0 \otimes_{\mathbb{k}} B \square_H H$ have a right H -comodule structure as follows:

$$\rho_1 : A \square_H H \otimes_{\mathbb{k}} \eta_0 \rightarrow A \square_H H \otimes_{\mathbb{k}} \eta_0 \otimes_{\mathbb{k}} H, \quad \rho_2 : \eta_0 \otimes_{\mathbb{k}} B \square_H H \rightarrow \eta_0 \otimes_{\mathbb{k}} B \square_H H \otimes_{\mathbb{k}} H,$$

$$\rho_1(a \otimes h \otimes 1) = a \otimes h_{(1)} \otimes 1 \otimes h_{(2)}, \quad \rho_2(1 \otimes b \otimes h) = 1 \otimes b \otimes h_{(1)} \otimes h_{(2)},$$

for all $a \otimes h \in A \square_H H$, $b \otimes h \in B \square_H H$. With these structures, the maps $\iota, \rho \otimes \text{id}, \text{id} \otimes \varepsilon$ and π are comodule morphisms. Hence, it is enough to prove that η_H is a right H -comodule map. First note that

$$\rho_1 = (\text{id} \otimes \eta_{H^t})(\xi_{H, H^t}^A \otimes \text{id}_{\eta_0})(\text{id} \otimes \Delta \otimes \text{id}), \quad \rho_2 = (\text{id}_{\eta_0} \otimes \xi_{H, H^t}^B)(\text{id} \otimes \text{id} \otimes \Delta).$$

Now, we have

$$\begin{aligned} (\eta_H \otimes \text{id}_H) \rho_1 &= (\eta_H \otimes \text{id}_H)(\text{id} \otimes \eta_{H^t})(\xi_{H, H^t}^A \otimes \text{id}_{\eta_0})(\text{id} \otimes \Delta \otimes \text{id}) \\ &= (\text{id}_{\eta_0} \otimes \xi_{H, H^t}^B) \eta_{H \otimes_{\mathbb{k}} H^t}(\text{id} \otimes \Delta \otimes \text{id}) \\ &= (\text{id}_{\eta_0} \otimes \xi_{H, H^t}^B)(\text{id} \otimes \text{id} \otimes \Delta) \eta_H = \rho_2 \eta_H. \end{aligned}$$

The second equality follows from (11), and the third one follows from the naturality of η since $\Delta : H \rightarrow H \otimes_{\mathbb{k}} H^t$ is a left H -comodule map. \square

Claim 4.8 $f : A^g \rightarrow B$ is a left H -comodule map.

Proof of Claim. If $\lambda : C \rightarrow H \otimes_{\mathbb{k}} C$ is a left H -comodule and $g \in G(H)$, define the left H -comodules $C^{(g)}$ and ${}^{(g)}C$ as follows. As vector spaces $C^{(g)} = {}^{(g)}C = C$, the comodule structures $\lambda^{(g)} : C^{(g)} \rightarrow H \otimes_{\mathbb{k}} C^{(g)}$, ${}^{(g)}\lambda : {}^{(g)}C \rightarrow H \otimes_{\mathbb{k}} {}^{(g)}C$ are defined by

$$\lambda^{(g)}(c) = c_{(-1)}g \otimes c_{(0)}, \quad {}^{(g)}\lambda(c) = gc_{(-1)} \otimes c_{(0)},$$

for all $c \in C$. Note that $f : A^g \rightarrow B$ is a left H -comodule map if and only if $f : A^{(g)} \rightarrow {}^{(g)}B$ is a left H -comodule map. It is enough to observe that $\iota : A^{(g)} \rightarrow A \otimes_{\mathbb{k}} \eta_0$ and $\pi : \eta_0 \otimes_{\mathbb{k}} B \rightarrow {}^{(g)}B$ are comodule morphisms. \square

Define $\text{InnbiGal}(H)$ as the set of isomorphism classes of (H, H) -biGalois objects A such that $A \sim H$.

Corollary 4.9 *There is an exact sequence of groups*

$$1 \rightarrow \text{InnbiGal}(H) \rightarrow \text{BiGal}(H) \rightarrow \text{BrPic}(\text{Rep}(H)).$$

Proof. Define the map $\phi : \text{BiGal}(H) \rightarrow \text{BrPic}(\text{Rep}(H))$, $\phi([A]) = [{}_A \mathcal{M}]$, for any isomorphism class $[A] \in \text{BiGal}(H)$. Here ${}_A \mathcal{M}$ denotes the equivalence class of the bimodule category ${}_A \mathcal{M}$. By Lemma 4.1 it follows that ϕ is well-defined and by Theorem 2.7 it is a group map. If $\phi([A])$ is the trivial element in $\text{BrPic}(\text{Rep}(H))$, by Theorem 4.5 it follows that $A \sim H$. \square

Remark 4.10 The above exact sequence can be thought of as an analogue of the sequence studied in [5]. It would be interesting to give an interpretation of the results obtained in [5] in this context.

Corollary 4.11 *If H^* is a quasi-triangular Hopf algebra, there is an injective group homomorphism $\text{BiGal}(H) \rightarrow \text{BrPic}(\text{Rep}(H))$.*

Proof. Let $A \in \text{InnbiGal}(H)$. It follows from Theorem 4.5 that there is a pseudo-natural isomorphism between the monoidal functors \mathcal{F}_A and \mathcal{F}_H . From Lemma 3.1 it follows that there is a natural monoidal isomorphism between \mathcal{F}_A and \mathcal{F}_H . This implies that $A \simeq H$ as biGalois objects. \square

5 Families of invertible $\text{Rep}(T_q)$ -bimodule categories

Let $n \geq 2$ be a natural number and q a n -th primitive root of unity. The Taft algebra is

$$T_q = \mathbb{k}\langle g, x | g^n = 1, x^n = 0, gx = q xg \rangle.$$

The structure of a Hopf algebra on T_q is such that g is group-like, x is $(1, g)$ -primitive, that is, $\Delta(x) = x \otimes 1 + g \otimes x$ with $S(x) = -g^{-1}x$. When $n = 2$ note that we recover Sweedler's Hopf algebra H_4 . The Taft algebra is isomorphic to a Radford biproduct

$$T_q \cong k[x]/(x^n) \# \mathbb{k}\mathbb{Z}_n \quad (20)$$

(send $G \mapsto 1 \otimes g$ and $X \mapsto x \otimes 1$), where $g \cdot x = qx$. The following technical result will be needed later.

Lemma 5.1 *There is a Hopf algebra isomorphism $\phi : T_{q^{-1}} \xrightarrow{\cong} T_q^{\text{cop}}$.*

Proof. Let us assume that T_q is generated by elements g, x such that $g^n = 1, x^n = 0, gx = qxg$ and

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x,$$

and $T_{q^{-1}}$ is generated by elements g, y such that $g^n = 1, y^n = 0, g^{-1}y = q^{-1}yg^{-1}$ and the coproduct is determined by

$$\Delta(g) = g \otimes g, \quad \Delta(y) = y \otimes 1 + g^{-1} \otimes y.$$

(Strictly speaking, in $T_{q^{-1}}$ we should take g^{-1} for a generator rather than g , but all the relations remain the same.) The algebra map $\phi : T_{q^{-1}} \rightarrow T_q^{\text{cop}}$ determined by $\phi(g) = g$, $\phi(y) = xg^{-1}$, is a well-defined Hopf algebra isomorphism. \square

As we are interested in finding invertible $\text{Rep}(T_q)$ -bimodule categories, which are left $\text{Rep}(T_q \otimes T_q^{\text{cop}})$ -module categories, from now on we shall consider the Hopf algebra $H = T_q \otimes T_{q^{-1}}$.

Let V_1 and V_2 be the one dimensional vector spaces spanned by x and y respectively, $\mathbb{Z}_n = \langle g \rangle = \langle g^{-1} \rangle$ with $g^n = 1$ and let $G = \mathbb{Z}_n \times \mathbb{Z}_n = \langle g \rangle \times \langle g \rangle$. The vector spaces V_1 and V_2 are G -modules via

$$(g^i, g^j) \cdot x = g^i \cdot x = q^i x \quad \text{and} \quad (g^i, g^j) \cdot y = g^j \cdot y = q^j y.$$

The algebra H is generated by the elements $\{f \in G, x, y\}$ subject to relations

$$x^n = 0 = y^n, \quad xy = yx, \quad fx = (f \cdot x)f, \quad fy = (f \cdot y)f.$$

Its Hopf algebra structure is given by

$$\Delta(x) = x \otimes 1 + (g, 1) \otimes x, \quad \Delta(y) = y \otimes 1 + (1, g^{-1}) \otimes y, \quad \Delta(f) = f \otimes f.$$

In other words, $H = \mathfrak{B}(V) \# \mathbb{k}G$ where $\mathfrak{B}(V)$ is the Nichols algebra of the Yetter-Drinfeld module $V = V_1 \oplus V_2$ over $\mathbb{k}G$. The coaction $\delta : V_1 \oplus V_2 \rightarrow \mathbb{k}G \otimes_{\mathbb{k}} (V_1 \oplus V_2)$ is given by

$$\delta(v) = (g, 1) \otimes v, \quad \delta(w) = (1, g^{-1}) \otimes w,$$

for all $v \in V_1, w \in V_2$. Let us define a new Hopf algebra that will be used later. If $\chi_1, \chi_2 : G \rightarrow \mathbb{k}$ are characters then V has a new action of G as follows. For any $f \in G, v \in V_1, w \in V_2$

$$f \triangleright v = \chi_1(f) f \cdot v, \quad f \triangleright w = \chi_2(f) f \cdot w.$$

Let χ_1, χ_2 be characters such that $\chi_1(1, g^{-1})\chi_2(g, 1) = 1$. In this case V with the new action and the same coaction is a Yetter-Drinfeld module over $\mathbb{k}G$ that we shall denote by $V_{(\chi_1, \chi_2)}$. Observe that V and also $V_{(\chi_1, \chi_2)}$ are quantum linear spaces, see [2].

Definition 5.2 *If $\chi_1, \chi_2 : G \rightarrow \mathbb{k}$ are characters such that $\chi_1(1, g^{-1})\chi_2(g, 1) = 1$ we denote $H_{(\chi_1, \chi_2)} = \mathfrak{B}(V_{(\chi_1, \chi_2)}) \# \mathbb{k}G$.*

The algebra $H_{(\chi_1, \chi_2)}$ is generated by elements $\{f \in G, x, y\}$ subject to relations

$$x^n = 0 = y^n, \quad xy = \chi_2(g, 1) yx, \quad fx = (f \triangleright x)f, \quad fy = (f \triangleright y)f.$$

Its coproduct is the same as the coproduct of H .

5.1 Twisting $T_q \otimes T_{q^{-1}}$

We will next investigate the Hopf algebra $(T_q \otimes T_{q^{-1}})^{[\sigma]}$ for Hopf 2-cocycles σ obtained as in Lemma 2.1. Let $\psi \in Z^2(G, \mathbb{k}^\times)$. We define $\chi_1, \chi_2 : G \rightarrow k^\times$ characters on G , via

$$\chi_1(f) = \frac{\psi(f, (g, 1))}{\psi((g, 1), f)} \quad \text{and} \quad \chi_2(f) = \frac{\psi(f, (1, g^{-1}))}{\psi((1, g^{-1}), f)}. \quad (21)$$

The proof of the following result is straightforward.

Proposition 5.3 *Assume $\psi \in Z^2(G, k^\times)$ is a 2-cocycle. Let $\sigma : H \otimes H \rightarrow \mathbb{k}$ be a 2-cocycle coming from ψ as in Lemma 2.1 and χ_1, χ_2 characters in G defined in (21). There is an isomorphism of Hopf algebras $H^{[\sigma]} \simeq H_{(\chi_1, \chi_2)}$.* \square

5.2 Homogeneous coideal subalgebras in Taft Hopf algebra

Due to [23, Theorem 6.1] any coideal subalgebra of a finite-dimensional Hopf algebra H is an H -simple comodule algebra. Its *lifting* will also be of that type and thus it will determine an exact indecomposable $\text{Rep}(H)$ -bimodule category. Any exact indecomposable module category emerges in this way. This is why a fundamental piece of information needed to compute exact $\text{Rep}(T_q)$ -bimodule categories is the classification of its coideal subalgebras. This is the main goal of this section. As before, we set $H = T_q \otimes T_{q^{-1}}$.

Note that $H(1) = (V_1 \oplus V_2) \otimes \mathbb{k}G$. For $(v_1, v_2) = (\alpha x, \beta y) \in V_1 \oplus V_2$, with $\alpha, \beta \in \mathbb{k}$, we will denote

$$[(v_1, v_2)] = v_1 + v_2(g, g) \in H(1) \quad \text{and} \quad [\widetilde{(v_1, v_2)}] = v_2 + v_1(g^{-1}, g^{-1}) \in H(1).$$

Remark 5.4 The following holds:

$$[(v_1, v_2)]^n = [\widetilde{(v_1, v_2)}]^n = 0 \quad (22)$$

$$\Delta([(v_1, v_2)]) = v_1 \otimes 1 + v_2(g, g) \otimes (g, g) + (g, 1) \otimes [(v_1, v_2)] \quad (23)$$

$$\Delta([\widetilde{(v_1, v_2)}]) = v_2 \otimes 1 + v_1(g^{-1}, g^{-1}) \otimes (g^{-1}, g^{-1}) + (1, g^{-1}) \otimes [\widetilde{(v_1, v_2)}]. \quad (24)$$

Observe that if K is a homogeneous left coideal subalgebra of H and $[(v_1, v_2)] \in K$ or $[\widetilde{(v_1, v_2)}] \in K$ where some v_i is not null then, since both (23) and (24) are elements in $H(0) \otimes K(1) \oplus H(1) \otimes K(0)$, it follows that $(g, g) \in K(0)$.

Definition 5.5 *A coideal subalgebra datum is a collection (W^1, W^2, W^3, F) such that*

1. $W = W^1 \oplus W^2 \oplus W^3$ is a subspace of $V_1 \oplus V_2$ such that

$$W \cap V_1 = W^1, \quad W \cap V_2 = W^2;$$

2. $W^3 \subseteq V_1 \oplus V_2$ is a subspace such that

$$W^3 \cap W^1 \oplus W^2 = 0, \quad W^3 \cap V_1 = 0 = W^3 \cap V_2;$$

3. $F \subseteq G$ is a subgroup that leaves invariant all subspaces W^i , $i = 1, 2, 3$;

4. if $W^3 \neq 0$ then $(g, g) \in F$;

We denote by $C(W^1, W^2, W^3, F)$ the subalgebra of H generated by $\mathbb{k}F$ and elements in $W^1 \oplus W^2$ and $\{[w], \widetilde{[w]} : w \in W^3\}$.

If $\chi_1, \chi_2 : G \rightarrow \mathbb{k}$ are characters such that $\chi_1(1, g^{-1})\chi_2(g, 1) = 1$ and (W^1, W^2, W^3, F) is a coideal subalgebra datum, we shall denote by $C_{(\chi_1, \chi_2)}(W^1, W^2, W^3, F)$ the subalgebra of $H_{(\chi_1, \chi_2)}$ generated by $\mathbb{k}F$ and elements in $W^1 \oplus W^2$ and $\{[w], \widetilde{[w]} : w \in W^3\}$.

Remark 5.6 If (W^1, W^2, W^3, F) is a coideal subalgebra datum then we conclude that if $W^3 \neq 0$ then $W^1 = W^2 = 0$. Indeed, suppose that $W^3 \neq 0$ and $W^1 \neq 0$. Then $W^1 \oplus W^3 = W$ and $(W^1 \oplus W^3) \cap V_2 \neq 0$, since V_1 has dimension 1. This implies that $W^2 \neq 0$, but if $W^1 \neq 0$ and $W^2 \neq 0$, then it must be $W^3 = 0$, contradiction.

Lemma 5.7 *The algebra $C(W^1, W^2, W^3, F)$ (resp. $C_{(\chi_1, \chi_2)}(W^1, W^2, W^3, F)$) is a homogeneous left coideal subalgebra of H (resp. $H_{(\chi_1, \chi_2)}$). \square*

Theorem 5.8 *Any homogeneous left coideal subalgebra $K = \bigoplus_{i=0}^m K(i)$ in H (resp. $H_{(\chi_1, \chi_2)}$) is of the form $K = C(W^1, W^2, W^3, F)$ (resp. $C_{(\chi_1, \chi_2)}(W^1, W^2, W^3, F)$), for some coideal subalgebra datum (W^1, W^2, W^3, F) .*

Proof. We shall assume that χ_1, χ_2 are trivial, since the proof for non-trivial characters is completely analogous.

Given that $K(0) \subseteq \mathbb{k}G$ is a left coideal subalgebra, it is $K(0) = \mathbb{k}F$ for some subgroup $F \subseteq G$. If $K(1) = 0$, then $K = \mathbb{k}F$. Indeed, if $x \in K(2)$ then $\Delta(x) \in H(0) \otimes K(2) \oplus H(2) \otimes K(0)$, therefore $x \in H(0) \oplus H(1)$, and since $(H(0) \oplus H(1)) \cap H(2) = 0$, it follows $x = 0$. Similarly, one proves that $K(n) = 0$ for all n .

Suppose that $K(1) \neq 0$. The vector space $K(1)$ is a $\mathbb{k}G$ -subcomodule of $(V_1 \oplus V_2) \otimes \mathbb{k}G$ via

$$(\pi \otimes \text{id})\Delta : K(1) \rightarrow \mathbb{k}G \otimes K(1),$$

where $\pi : H \rightarrow \mathbb{k}G$ is the canonical projection. We may write $K(1) = \bigoplus_{f \in G} K(1)_f$ where $K(1)_f = \{x \in K(1) \mid (\pi \otimes \text{id})\Delta(x) = f \otimes x\}$. Then it is

$$K(1)_f \subseteq V_1 \otimes \mathbb{k}\langle(g^{-1}, 1)f\rangle \oplus V_2 \otimes \mathbb{k}\langle(1, g)f\rangle.$$

In particular we have:

$$K(1)_{(g,1)} = W^1 \oplus \widetilde{W}^2(g, g) \oplus U^3 \quad K(1)_{(1,g^{-1})} = W^2 \oplus \widetilde{W}^1(g^{-1}, g^{-1}) \oplus \widetilde{U}^3.$$

Here W^1 is the intersection of $K(1)_{(g,1)}$ with V_1 , $\widetilde{W}^2(g, g)$ is the intersection of $K(1)_{(g,1)}$ with $V_2 \otimes \mathbb{k}\langle(g, g)\rangle$, and U^3 is a direct complement. Concretely, U^3 is a subspace of $V_1 \oplus V_2 \otimes \mathbb{k}\langle(g, g)\rangle$ consisting of elements of the form $[w]$, where $w \in W^3$ and $W^3 \subseteq V_1 \oplus V_2$. Given that $U^3 \cap W^1 \oplus \widetilde{W}^2(g, g) = 0$, it follows that $W^3 \cap W^1 \oplus \widetilde{W}^2 = 0$.

Analogously, for $K(1)_{(1,g^{-1})}$ we have that W^2 is the intersection of $K(1)_{(1,g^{-1})}$ with V_2 , $\widetilde{W}^1(g^{-1}, g^{-1})$ is the intersection of $K(1)_{(1,g^{-1})}$ with $V_1 \otimes \mathbb{k}\langle(g^{-1}, g^{-1})\rangle$, and \widetilde{U}^3 is a direct complement. The elements of \widetilde{U}^3 are of the form $[\widetilde{w}]$, where $w \in \widetilde{W}^3$ and $\widetilde{W}^3 \subseteq V_1 \oplus V_2$. We next prove the following two results:

Claim 5.9 *If any of $\widetilde{W}^1, \widetilde{W}^2, \widetilde{W}^3$ or W^3 is different from 0, then $(g, g) \in F$. If $\widetilde{W}^1 \neq 0$, then $\widetilde{W}^1 = W^1$. If $\widetilde{W}^2 \neq 0$, then $\widetilde{W}^2 = W^2$.*

Proof. Take $0 \neq (v_1, v_2) \in W^3$. Then $0 \neq [(v_1, v_2)] \in U^3$ and by Remark 5.4 $(g, g) \in F$. The proof is analogous if $\widetilde{W}^1, \widetilde{W}^2$ or \widetilde{W}^3 are different from zero. For the second claim take $w \in \widetilde{W}^1$. Then $w(g^{-1}, g^{-1}) \in K(1)_{(1,g^{-1})}$ and $w \in K(1)_{(g,1)}$ and it must be $w \in W^1$. Similarly, one proves the other inclusion and we get $\widetilde{W}^1 = W^1$. The other equality is proven analogously. \square

Claim 5.10 $K(1) = W^1F \oplus W^2F \oplus U^3F$.

Proof. Take $f \in G$ and $0 \neq x \in K(1)_f$. Then for some $v_1 \in V_1$ and $v_2 \in V_2$ it is

$$x = v_1(g^{-1}, 1)f + v_2(1, g)f$$

and

$$\Delta(x) = v_1(g^{-1}, 1)f \otimes (g^{-1}, 1)f + v_2(1, g)f \otimes (1, g)f + f \otimes x$$

is an element in $H(0) \otimes K(1) \oplus H(1) \otimes K(0)$. If $v_1 \neq 0$, then $(g^{-1}, 1)f \in F$ and hence $xf^{-1}(g, 1) \in K(1)_{(g,1)}$ and $x \in K(1)_{(g,1)}F \subseteq W^1F \oplus \widetilde{W}^2(g, g)F \oplus U^3F \subseteq W^1F \oplus W^2F \oplus U^3F$ - the latter inclusion is due to Claim 5.9. If $v_1 = 0$, then $v_2 \neq 0$ and it follows $(1, g)f \in F$. Thus $xf^{-1}(1, g^{-1}) \in K(1)_{(1,g^{-1})}$. If $\widetilde{W}^1 = \widetilde{U}^3 = 0$, then $xf^{-1}(1, g^{-1}) \in W^2$ and $x \in W^2F$ and the claim follows. If any of \widetilde{W}^1 and \widetilde{U}^3 is not zero, then $(g, g) \in F$. But then $(g^{-1}, 1)f = (g^{-1}, g^{-1})(1, g)f \in F$ and thus $xf^{-1}(g, 1) \in K(1)_{(g,1)}$. It follows $x \in K(1)_{(g,1)}F$, which we already have proved is a subspace of $W^1F \oplus W^2F \oplus U^3F$. \square

To finish the proof of the Theorem we will prove that K is generated as an algebra by $K(0)$ and $K(1)$, which yields $K = C(W^1, W^2, W^3, F)$. Let $B = \{b_i\}$ be a basis of $V_1 \oplus V_2$. For any $v_1 \in V_1$ and $v_2 \in V_2$ it is $(v_1, 0) = \sum_i \alpha_i b_i$ and $(0, v_2) = \sum_i \beta_i b_i$ for

some $\alpha_i, \beta_i \in k$. Then $v_1 = \sum_i \alpha_i [b_i]$ and $v_2 = \sum_i \beta_i [b_i](g^{-1}, g^{-1})$. So we have that H is generated as an algebra by the set

$$\{[b_i], f : b_i \in B, f \in G\}.$$

Now, let $\{b_1, \dots, b_r\}$ be a basis of $W = W^1 \oplus W^2 \oplus W^3$ and extend it to a basis $\{b_1, \dots, b_t\}$ of $V_1 \oplus V_2$ with $r \leq t$. For $n > 1$ an arbitrary $x \in K(n)$ has the form

$$x = \sum_{\substack{s_j \in \{0,1\} \\ f_i \in G}} \alpha_{s_1, \dots, s_t, i} [b_1]^{s_1} [b_2]^{s_2} \dots [b_t]^{s_t} f_i$$

for some $\alpha_{s_1, \dots, s_t, i} \in k$, where $s_1 + \dots + s_t = n$. Let $p : H \rightarrow H(1)$ be the canonical projection. Then

$$(\text{id} \otimes p)\Delta(x) = \sum_l \sum_{\substack{s_j \in \{0,1\} \\ f_i \in G}} \alpha_{s_1, \dots, s_t, i} h_{s_1, \dots, s_t, i, l} \otimes [b_l] f_i$$

for some $0 \neq h_{s_1, \dots, s_t, i, l} \in H(n-1)$ is an element in $H(n-1) \otimes K(1)$. So for all $l > r$ with $s_l = 1$ it must be $\alpha_{s_1, \dots, s_t, i} = 0$ and $K(n)$ is generated as an algebra by $K(1)$. \square

Remark 5.11 A coideal subalgebra datum depends whether it is on H or $H_{(\chi_1, \chi_2)}$. Since V_1, V_2 are 1-dimensional vector spaces we can give a description of all possible coideal subalgebra data. Let (W^1, W^2, W^3, F) be a coideal subalgebra datum (either for H or $H_{(\chi_1, \chi_2)}$). Then W^3 is either null or 1-dimensional. If $W^3 \neq 0$ then, using Remark 5.6 we get that $W^1 = W^2 = 0$ and

$$(W^1, W^2, W^3) = (0, 0, \langle \xi x + y \rangle_{\mathbb{k}})$$

for some $0 \neq \xi \in \mathbb{k}$. We call this coideal subalgebra datum of *type* ξ .

If $W^3 = 0$ then

$$(W^1, W^2, W^3) = (\langle \delta_1 x \rangle_{\mathbb{k}}, \langle \delta_2 y \rangle_{\mathbb{k}}, 0)$$

for some $\delta_1, \delta_2 \in \{0, 1\}$. We call this coideal subalgebra datum of *type* (δ_1, δ_2) . We have already observed that if $W^3 \neq 0$, then $(g, g) \in F$.

Assume (W^1, W^2, W^3, F) is a coideal subalgebra datum for $H_{(\chi_1, \chi_2)}$. Let $f = (g^i, g^j) \in F$. Since F leaves invariant the subspace W^3 , then

$$f \cdot (\xi x + y) = \xi q^i \chi_1(f) x + q^j \chi_2(f) y \in \langle \xi x + y \rangle_{\mathbb{k}}.$$

Hence

$$q^i \chi_1(f) = q^j \chi_2(f) \tag{25}$$

for all $f \in F$. In particular, if $\chi_1 = \chi_2$, then $i = j$. Therefore F contains the cyclic group generated by (g, g) .

5.3 Families of $T_q \otimes T_{q^{-1}}$ -comodule algebras

We shall introduce families of non-equivalent right H -simple left H -comodule algebras and *a fortiori*, families of exact indecomposable $\text{Rep}(H)$ -module categories, where $H = T_q \otimes T_{q^{-1}}$. We shall define them by generators and relations extending the information from the coideal subalgebras from the previous section. It will turn out that the former families are liftings of the latter.

Definition 5.12 *Given a subgroup $F \subseteq \mathbb{Z}_n \times \mathbb{Z}_n$ we shall say that a 2-cocycle $\psi \in Z^2(\mathbb{Z}_n \times \mathbb{Z}_n, \mathbb{k}^\times)$ is compatible with F if*

$$q^i \frac{\psi((g, 1), f)}{\psi(f, (g, 1))} = q^j \frac{\psi((1, g^{-1}), f)}{\psi(f, (1, g^{-1}))} \quad (26)$$

for any $f = (g^i, g^j) \in F$. We shall say that a 2-cocycle $\psi \in Z^2(F, \mathbb{k}^\times)$ is compatible with F if the corestriction (see [4, p. 81]) of ψ in $Z^2(G, \mathbb{k}^\times)$ satisfies (26).

Remark 5.13 Equation (26) is obtained by replacing the values of χ_1, χ_2 given in (21), using ψ^{-1} , in equation (25).

Let us introduce five families of left H -comodule algebras.

- Let $F \subseteq \mathbb{Z}_n \times \mathbb{Z}_n$ be a subgroup such that $(g, g) \in F$, $\psi \in Z^2(F, \mathbb{k}^\times)$ a 2-cocycle compatible with F , $\xi, \mu \in \mathbb{k}$ with $\xi \neq 0$. Set $\mathcal{L}(\xi, \mu, F, \psi)$ for the algebra generated by elements $\{w, e_f : f \in F\}$ subject to relations

$$w^n = \mu 1, \quad e_f e_{f'} = \psi(f, f') e_{ff'}, \quad e_f w = \tau_f w e_f.$$

Here $\tau_f = q^i$ if $f = (g^i, g^j)$. The left comodule structure $\lambda : \mathcal{L}(\xi, \mu, F, \psi) \rightarrow H \otimes_{\mathbb{k}} \mathcal{L}(\xi, \mu, F, \psi)$ is defined by

$$\lambda(e_f) = f \otimes e_f, \quad \lambda(w) = \xi x \otimes 1 + y(g, g) \otimes e_{(g, g)} + (g, 1) \otimes w.$$

- Let $a, b, \xi \in \mathbb{k}$, $F \subseteq G$ a subgroup, $\psi \in Z^2(F, \mathbb{k}^\times)$. Set $\mathcal{K}_{11}(a, b, \xi, F, \psi)$ for the algebra generated by elements $\{z, u, e_f : f \in F\}$ subject to relations

$$z^n = a 1, \quad u^n = b 1, \quad zu - uz = \xi e_{(g, g^{-1})}, \quad e_f e_{f'} = \psi(f, f') e_{ff'},$$

$$e_{(g^i, g^j)} z = q^i z e_{(g^i, g^j)}, \quad e_{(g^i, g^j)} u = q^j u e_{(g^i, g^j)} \quad \text{if } (g^i, g^j) \in F.$$

If $(g, g^{-1}) \notin F$ then $\xi = 0$. The coaction $\lambda : \mathcal{K}_{11}(a, b, \xi, F, \psi) \rightarrow H \otimes_{\mathbb{k}} \mathcal{K}_{11}(a, b, \xi, F, \psi)$ is defined by

$$\lambda(e_f) = f \otimes e_f, \quad \lambda(z) = x \otimes 1 + (g, 1) \otimes z, \quad \lambda(u) = y \otimes 1 + (1, g^{-1}) \otimes u.$$

- The algebra $\mathcal{K}_{01}(a, F, \psi)$ is the subcomodule algebra of $\mathcal{K}_{11}(a, b, \xi, F, \psi)$ generated by elements $\{z, e_f : f \in F\}$.
- The algebra $\mathcal{K}_{10}(b, F, \psi)$ is the subcomodule algebra of $\mathcal{K}_{11}(a, b, \xi, F, \psi)$ generated by elements $\{u, e_f : f \in F\}$.

- Let $F \subseteq \mathbb{Z}_n \times \mathbb{Z}_n$ be a subgroup, $\psi \in Z^2(F, \mathbb{k}^\times)$ a 2-cocycle then $\mathbb{k}_\psi F$ is the twisted group algebra.

Remark 5.14 The first family of comodule algebras is related to the coideal subalgebra datum of type ξ and the other four families are related to the coideal subalgebra datum of type $(1, 1)$, $(0, 1)$, $(1, 0)$ and $(0, 0)$ respectively.

Lemma 5.15 *The algebras $\mathcal{L}(\xi, \mu, F, \psi)$, $\mathcal{K}_{11}(a, b, \xi, F, \psi)$, $\mathcal{K}_{01}(a, F, \psi)$, $\mathcal{K}_{10}(b, F, \psi)$ are right H -simple left H -comodule algebras with trivial coinvariants.*

Proof. It is enough to note that

$$\mathbb{k}_\psi F = \mathcal{L}(\xi, \mu, F, \psi)_0 = \mathcal{K}_{11}(a, b, \xi, F, \psi)_0 = \mathcal{K}_{01}(a, F, \psi)_0 = \mathcal{K}_{10}(b, F, \psi)_0$$

and use [18, Prop. 4.4]. □

Let $\psi \in Z^2(\mathbb{Z}_n \times \mathbb{Z}_n, \mathbb{k}^\times)$ and $\sigma_\psi : H \otimes_{\mathbb{k}} H \rightarrow \mathbb{k}$ be the associated Hopf 2-cocycle. Let χ_1, χ_2 be the characters in $\mathbb{Z}_n \times \mathbb{Z}_n$ defined in (21) and let (W^1, W^2, W^3, F) be a coideal subalgebra datum for $H_{(\chi_1, \chi_2)}$.

Lemma 5.16 *If $W^3 = 0$ there is an isomorphism of comodule algebras*

$$C_{(\chi_1, \chi_2)}(W^1, W^2, 0, F)_{\sigma_\psi^{-1}} \simeq \mathcal{K}_{ij}(0, 0, 0, F, \psi^{-1})$$

for some $i, j \in \{0, 1\}$. If $W^3 \neq 0$ then $C_{(\chi_1, \chi_2)}(0, 0, W^3, F)_{\sigma_\psi^{-1}} \simeq \mathcal{L}(\xi, 0, F, \psi^{-1})$ for some $\xi \in \mathbb{k}^\times$. □

5.4 Classification of exact module categories over $\text{Rep}(T_q \otimes T_{q^{-1}})$

In this section we give a classification of exact indecomposable $\text{Rep}(T_q)$ -bimodule categories. This is a new result, interesting in itself.

Theorem 5.17 *Let \mathcal{M} be an exact indecomposable $\text{Rep}(T_q)$ -bimodule category then \mathcal{M} is equivalent to one of the following categories:*

- $\mathbb{k}_\psi F \mathcal{M}$ for some subgroup $F \subseteq G$ and $\psi \in Z^2(F, \mathbb{k}^\times)$;
- $\mathcal{L}(\xi, \mu, F, \psi) \mathcal{M}$ for some subgroup $F \subseteq G$ such that $(g, g) \in F$ and $\psi \in Z^2(F, \mathbb{k}^\times)$ is compatible with F , $\xi, \mu \in \mathbb{k}$ with $\xi \neq 0$;
- $\mathcal{K}_{11}(a, b, \xi, F, \psi) \mathcal{M}$ for some $a, b, \xi \in \mathbb{k}$, $F \subseteq G$ a subgroup, $\psi \in Z^2(F, \mathbb{k}^\times)$;
- $\mathcal{K}_{01}(a, F, \psi) \mathcal{M}$ for some $a \in \mathbb{k}$, $F \subseteq G$ a subgroup, $\psi \in Z^2(F, \mathbb{k}^\times)$;
- $\mathcal{K}_{10}(b, F, \psi) \mathcal{M}$ for some $a \in \mathbb{k}$, $F \subseteq G$ a subgroup, $\psi \in Z^2(F, \mathbb{k}^\times)$.

Proof. By Lemma 5.15 all module categories listed above are exact indecomposable. Let \mathcal{M} be an indecomposable exact $\text{Rep}(T_q)$ -bimodule category, then it is an indecomposable exact $\text{Rep}(H)$ -module category. By [1, Thm 3.3] there exists a right H -simple left comodule algebra with trivial coinvariants (A, λ) , $\lambda : A \rightarrow H \otimes_{\mathbb{k}} A$, such that $\mathcal{M} = {}_A \mathcal{M}$ as $\text{Rep}(H)$ -modules. Since H is coradically graded then $\text{gr } A$ is a right H -simple left comodule algebra also with trivial coinvariants. Thus, there exists a subgroup $F \subseteq G$ and $\psi \in Z^2(F, \mathbb{k}^\times)$ such that $\text{gr } A_0 = \mathbb{k}_\psi F$.

Abusing the notation we shall denote by $\psi \in Z^2(G, \mathbb{k}^\times)$ the 2-cocycle such that restricted to F it equals ψ . Since $(\text{gr } A)_{\sigma_\psi^{-1}}$ is a Loewy-graded comodule algebra in $H^{[\sigma_\psi^{-1}]}$, it follows from [20, Lemma 5.5] that $(\text{gr } A)_{\sigma_\psi^{-1}}$ is isomorphic to a homogeneous coideal subalgebra of $H^{[\sigma_\psi^{-1}]}$. Let χ_1, χ_2 be the characters in G defined in (21) using ψ^{-1} . Then $H^{[\sigma_\psi^{-1}]} = H_{(\chi_1, \chi_2)}$ and by Theorem 5.8 $(\text{gr } A)_{\sigma_\psi^{-1}} = C_{(\chi_1, \chi_2)}(W^1, W^2, W^3, F)$ for some coideal subalgebra datum (W^1, W^2, W^3, F) . Then $\text{gr } A = C_{(\chi_1, \chi_2)}(W^1, W^2, W^3, F)_{\sigma_\psi}$ and by Lemma 5.16 there are two options: when $W^3 = 0$ and $W^3 \neq 0$.

We shall only analyze the case when $W^3 \neq 0$, the other case is done similarly. In this case, by Lemma 5.16, $\text{gr } A = \mathcal{L}(\xi, 0, F, \psi)$.

Claim 5.18 *There exists an element $w \in A$ such that*

$$\lambda(w) = \xi x \otimes 1 + y(g, g) \otimes e_{(g, g)} + (g, 1) \otimes w. \quad (27)$$

Proof. Since $\text{gr } A = \mathcal{L}(\xi, 0, F, \psi)$ there exists an element $\bar{w} \in A_1/A_0$ such that

$$\bar{\lambda}(\bar{w}) = \xi x \otimes 1 + y(g, g) \otimes e_{(g, g)} + (g, 1) \otimes \bar{w}.$$

Here $\bar{\lambda} : \text{gr } A \rightarrow H \otimes_{\mathbb{k}} \text{gr } A$ is the coaction induced from λ , see for example [18, Section 4]. Let us denote by $w' \in A_1$ a representative element in the class of \bar{w} . By the definition of the induced coaction $\bar{\lambda}$ there exists a map $\lambda_1 : \text{gr } A \rightarrow \mathbb{k}G \otimes_{\mathbb{k}} \mathbb{k}_\psi F$ such that

$$\lambda(w') = \xi x \otimes 1 + y(g, g) \otimes e_{(g, g)} + (g, 1) \otimes w' + \lambda_1(w').$$

Set $\lambda_1(w') = \sum_{h \in G, f \in F} a_{h, f} h \otimes e_f$, for some $a_{h, f} \in \mathbb{k}$. By the coassociativity of λ we get that $(g, 1) \otimes \lambda_1(w') + (\text{id} \otimes \lambda)\lambda_1(w') = (\Delta \otimes \text{id})\lambda_1(w')$, whence

$$\lambda_1(w') = \sum_{f \in F, f \neq (g, 1)} \beta_f ((g, 1) - f) \otimes e_f.$$

Then, if $z = \sum_{f \in F, f \neq (g, 1)} \beta_f e_f \in A_0$ we get that $\lambda_1(w') = (g, 1) \otimes z - \lambda(z)$. The element $w = w' + z$ satisfies equation (27). \square

It is not difficult to prove that we can choose one $w \in A$ which also satisfies $e_f w = (f \cdot w)e_f$, where the action $\cdot : F \times W^3 \rightarrow W^3$ is the restriction of the action of G on $V_1 \oplus V_2$. The set $\{fw^i : f \in F, 0 \leq i < n\}$ is a basis for A . Since

$$\lambda(w^n) = 1 \otimes w^n,$$

there exists $\mu \in \mathbb{k}$ such that $w^n = \mu 1$. Hence, there is a projection $\mathcal{L}(\xi, \mu, F, \psi) \rightarrow A$ that must be an isomorphism since both algebras have the same dimension. \square

We shall analyze when module categories listed in Theorem 5.17 are equivalent.

Proposition 5.19 *The following statements hold:*

1. ${}_{\mathbb{k}_\psi F} \mathcal{M} \simeq {}_{\mathbb{k}_{\psi'} F'} \mathcal{M}$ if and only if $F = F'$ and $\psi = \psi'$ in $H^2(F, \mathbb{k}^\times)$;

2. $\mathcal{L}(\xi, \mu, F, \psi) \mathcal{M} \simeq \mathcal{L}(\xi', \mu', F', \psi') \mathcal{M}$ if and only if $\xi = q^i \xi'$, $\mu = \mu'$ for some $i \in \mathbb{N}$ and $F = F'$ and $\psi = \psi'$ in $H^2(F, \mathbb{k}^\times)$;
3. $\kappa_{11}(a, b, \xi, F, \psi) \mathcal{M} \simeq \kappa_{11}(a', b', \xi', F', \psi') \mathcal{M}$ if and only if $(a, b, \xi, F, \psi) = (a', b', \xi', F', \psi')$;
4. for any $(i, j) \in \{(0, 1), (1, 0)\}$ $\kappa_{ij}(a, F, \psi) \mathcal{M} \simeq \kappa_{ij}(a', F', \psi') \mathcal{M}$ if and only if $(a, F, \psi) = (a', F', \psi')$.

Proof. We shall only prove (2). The proofs of the other statements are analogous. It is not difficult to prove that if there is an isomorphism of left H -comodule algebras $\mathcal{L}(\xi, \mu, F, \psi) \simeq \mathcal{L}(\xi', \mu', F', \psi')$, then $\xi = q^a \xi'$, $\mu = \mu'$, for some $a \in \mathbb{N}$ and $F = F'$, $\psi = \psi'$.

It follows from [15, Thm. 4.2] that $\mathcal{L}(\xi, \mu, F, \psi) \mathcal{M} \simeq \mathcal{L}(\xi', \mu', F', \psi') \mathcal{M}$ if and only if there exists $f \in G$ such that $\mathcal{L}(\xi, \mu, F, \psi) \simeq \mathcal{L}(\xi', \mu', F', \psi')^f$ as left H -comodule algebras, where the latter comodule algebra has the structure as in (15). One readily obtains that if $f \in G$ then $\mathcal{L}(\xi', \mu', F', \psi')^f = \mathcal{L}(q^i \xi', \mu', F', \psi')$ for some $i \in \mathbb{N}$. \square

One of the consequences of the above is the classification of T_q -biGalois objects. The classification was already obtained by Schauenburg, see [21].

Corollary 5.20 *If A is a T_q -biGalois object then $A \simeq \mathcal{L}(\xi, \mu, \text{diag}(G), 1)$ as T_q -bicomodule algebras for some $0 \neq \xi, \mu \in \mathbb{k}$.*

Proof. Let A be a T_q -biGalois object. Then A as a left $T_q \otimes_{\mathbb{k}} T_q^{\text{cop}}$ -comodule algebra has no non-trivial H -costable ideal. Indeed, let $I \subseteq A$ be an H -costable ideal. Thus I is a T_q -costable ideal and since A is biGalois, this means that $I = 0$ or $I = A$. Then A must be one of the algebras listed above. If we observe these algebras as T_q -bicomodules, it is easily seen that the only family of algebras that are T_q -biGalois is $\mathcal{L}(\xi, \mu, \text{diag}(G), 1)$ for some $0 \neq \xi, \mu \in \mathbb{k}$ (keep in mind that since T_q is finite-dimensional, every biGalois object is isomorphic to T_q as a bicomodule). \square

We shall denote $\mathcal{L}(\xi, \mu) = \mathcal{L}(\xi, \mu, \text{diag}(G), 1)$ for any $0 \neq \xi, \mu \in \mathbb{k}$. Two biGalois objects $\mathcal{L}(\xi, \mu), \mathcal{L}(\xi', \mu')$ are isomorphic if and only if $\mu = \mu', \xi = q^i \xi'$ for some $i \in \mathbb{N}$. Recall the definition of the equivalence relation \sim given in Definition 4.4. As in the proof of Proposition 5.19 we have that $\mathcal{L}(\xi, \mu) \sim \mathcal{L}(\xi', \mu')$ if and only if $\mu = \mu', \xi = q^i \xi'$ for some $i \in \mathbb{N}$. It is straightforward to see that $T_q \simeq \mathcal{L}(1, 0)$. We then have:

Corollary 5.21 *The subgroup $\text{InnbiGal}(T_q)$ from Corollary 4.9 is trivial and the map $\phi : \text{BiGal}(T_q) \rightarrow \text{BrPic}(\text{Rep}(T_q)), \phi([A]) = [{}_A \mathcal{M}]$ is a group embedding.*

5.5 Invertible $\text{Rep}(T_q)$ -bimodule categories

As a consequence of the above results we give an explicit family of invertible exact $\text{Rep}(T_q)$ -bimodule categories that form a subgroup inside $\text{BrPic}(\text{Rep}(T_q))$.

Define the group $\mathbb{k}^\times \ltimes \mathbb{k}^+$ with the underlying set $\mathbb{k} - \{0\} \times \mathbb{k}$ and product given by

$$(a, b) \cdot (c, d) = (ac, cb + d),$$

for any $(a, b), (c, d) \in \mathbb{k}^\times \times \mathbb{k}$. If \mathbb{G}_n denotes the subgroup of \mathbb{k}^\times of n -th roots of unity, then $\mathbb{k}^\times / \mathbb{G}_n \rtimes \mathbb{k}^+$ is a group with product

$$(\bar{a}, b) \cdot (\bar{c}, d) = (\bar{ac}, c^n b + d),$$

for any $(\bar{a}, b), (\bar{c}, d) \in \mathbb{k}^\times / \mathbb{G}_n \rtimes \mathbb{k}^+$. The map $\phi : \mathbb{k}^\times / \mathbb{G}_n \rtimes \mathbb{k}^+ \rightarrow \mathbb{k}^\times \rtimes \mathbb{k}^+$ given by $\phi(\bar{\xi}, \mu) = (\xi^n, \mu)$ is a group isomorphism. Schauenburg proved that there is a group isomorphism $\text{BiGal}(T_q) \simeq \mathbb{k}^\times \rtimes \mathbb{k}^+$, [21, Thm. 2.5]. We shall give another proof of this result, mainly for two reasons. Our description of biGalois objects is different from the one in [21] and we also want to show an explicit subgroup inside $\text{BrPic}(\text{Rep}(T_q))$.

Theorem 5.22 *Let $\xi, \mu, \xi', \mu' \in \mathbb{k}$, $\xi', \xi \neq 0$. There is an isomorphism of $T_q \otimes_{\mathbb{k}} T_{q^{-1}}$ -comodule algebras*

$$\mathcal{L}(\xi', \mu') \square_{T_q} \mathcal{L}(\xi, \mu) \simeq \mathcal{L}(\xi' \xi, \xi^n \mu' + \mu).$$

Proof. Recall that the left $T_q \otimes_{\mathbb{k}} T_{q^{-1}}$ -comodule structure on the cotensor product is given by (10). Let

$$\gamma : \mathcal{L}(\xi' \xi, \xi^n \mu' + \mu) \rightarrow \mathcal{L}(\xi', \mu') \square_{T_q} \mathcal{L}(\xi, \mu)$$

be the algebra map determined by

$$\gamma(w) = \xi w \otimes 1 + e_{(g,g)} \otimes w, \quad \gamma(e_f) = e_f \otimes e_f,$$

for all $f \in \text{diag}(G)$. Note that we are abusing the notation by denoting with the same name the generators of the algebras $\mathcal{L}(\xi, \mu)$, $\mathcal{L}(\xi', \mu')$ and $\mathcal{L}(\xi' \xi, \xi^n \mu' + \mu)$. To prove that γ is well-defined we have to verify that

$$\gamma(w^n) = (\xi^n \mu' + \mu) 1, \quad \gamma(e_{(g,g)} w) = q \gamma(w e_{(g,g)}).$$

This is done by a straightforward computation. Also, it can be readily proven that the image of γ is contained in $\mathcal{L}(\xi', \mu') \square_{T_q} \mathcal{L}(\xi, \mu)$ and that γ is an injective comodule morphism. To prove that γ is bijective, we shall prove that

$$\dim(\mathcal{L}(\xi', \mu') \square_{T_q} \mathcal{L}(\xi, \mu)) = \dim(\mathcal{L}(\xi' \xi, \xi^n \mu' + \mu)).$$

Since T_q is finite-dimensional, then any Hopf-Galois object is Cleft. This implies that $\mathcal{L}(\xi, \mu) \simeq T_q$ as a right and left T_q -comodules. Hence, there are linear isomorphisms

$$\mathcal{L}(\xi', \mu') \square_{T_q} \mathcal{L}(\xi, \mu) \simeq T_q \square_{T_q} T_q \simeq T_q.$$

Therefore $\dim(\mathcal{L}(\xi', \mu') \square_{T_q} \mathcal{L}(\xi, \mu)) = \dim(\mathcal{L}(\xi' \xi, \xi^n \mu' + \mu))$. □

As a consequence of the above Theorem there is a group isomorphism $\mathbb{k}^\times \rtimes \mathbb{k}^+ \rightarrow \text{BiGal}(T_q)$ given by $(\xi, \mu) \mapsto [\mathcal{L}(\phi^{-1}(\xi, \mu))]$.

Corollary 5.23 *There is an injective group homomorphism $\alpha : \mathbb{k}^\times \rtimes \mathbb{k}^+ \rightarrow \text{BrPic}(\text{Rep}(T_q))$ given by*

$$\alpha(\xi, \mu) = [\mathcal{L}(\phi^{-1}(\xi, \mu)) \mathcal{M}], \quad (\xi, \mu) \in \mathbb{k}^\times \rtimes \mathbb{k}^+.$$

□

Acknowledgments

The work of B.F. was supported by a postdoctoral fellowship of PEDECIBA, Uruguay. The work of M.M. was supported by Secyt (UNC), Mincyt (Córdoba) and CONICET. The authors want to thank C. Galindo and R. Heluani for helpful discussions. We also thank the referee for his many constructive comments.

References

- [1] N. ANDRUSKIEWITSCH and M. MOMBELLI. *On module categories over finite-dimensional Hopf algebras*. J. Algebra **314** (2007), 383–418.
- [2] N. ANDRUSKIEWITSCH and H.-J. SCHNEIDER, *Lifting of quantum linear spaces and pointed Hopf algebras of order p^3* , J. Algebra **209** (1998), 658–691.
- [3] J. BÉNABOU, *Introduction to bicategories*, Lecture notes in mathematics **47** (1967).
- [4] K. BROWN, *Cohomology of groups*, New York: Springer-Verlag, 1982. Print.
- [5] T. BARMEIER, J. FUCHS, I. RUNKEL, and C. SCHWEIGERT. *On the Rosenberg-Zelinsky sequence in abelian monoidal categories*. J. Reine Angew. Math. **642** (2010), 1–36.
- [6] F. BORCEUX, *Handbook of Categorical Algebra, Basic Category Theory (Encyclopedia of Mathematics and its Applications)*, Volume 1, Cambridge University Press 1994.
- [7] S. CAENEPEEL, S. CRIVEI, A. MARCUS and M. TAKEUCHI. *Morita equivalences induced by bimodules over Hopf-Galois extensions*. J. Algebra **314** (2007), 267–30
- [8] P. DELIGNE *Catègories tannakiennes*. The Grothendieck Festschrift, Vol. II, Progr. Math., **87**, Birkhäuser, Boston, MA, 1990, 111–195.
- [9] Y. DOI. *Unifying Hopf modules*. J. Algebra **153** (1992), 373–385.
- [10] A. DAVYDOV, L. KONG and I. RUNKEL. *Invertible defects and isomorphisms of rational CFTs*. ZMP-HH/10-13 Hamburger Beiträge zur Mathematik (2010).
- [11] P. ETINGOF, D. NIKSHYCH and V. OSTRIK. *Fusion categories and homotopy theory*. Quantum Topol. **1**, No. 3, (2010) 209–273.
- [12] P. ETINGOF and V. OSTRIK. *Finite tensor categories*. Mosc. Math. J. **4** No. 3 (2004), 627–654.
- [13] J. FUCHS, C. SCHWEIGERT and A. VALENTINO. *Bicategories for boundary conditions and for surface defects in 3-d TFT*, Comm. Math. Physics **321**, Issue 2, (2013) 543–575.
- [14] C. GALINDO. *Crossed product tensor categories*. J. Algebra **337** (2011) 233–252.

- [15] A. GARCÍA IGLESIAS and M. MOMBELLI. *Representations of the category of modules over pointed Hopf algebras over \mathbb{S}_3 and \mathbb{S}_4* . Pacific J. Math. vol **252** No. 2 (2011) 343–378.
- [16] J. GREENOUGH. *Monoidal 2-structure of Bimodule Categories*. J. Algebra **324** (2010) 1818–1859.
- [17] A. KITAEV and L. KONG. *Models for gapped boundaries and domain walls*. Comm. Math. Physics **313**, Issue 2, (2012) 351–373.
- [18] M. MOMBELLI. *Module categories over pointed Hopf algebras*. Math. Z. **266** No. 2 (2010) 319–344.
- [19] M. MOMBELLI. *On the tensor product of bimodule categories over Hopf algebras*. Abh. Math. Semin. Univ. Hamb. **82**, (2012) 173–192.
- [20] M. MOMBELLI. *The Brauer-Picard group of the representation category of finite supergroup algebras*. Preprint arXiv:1202.6238.
- [21] P. SCHAUENBURG. *Bi-Galois objects over the Taft algebras*. Israel J. Math. **115** (2000), 101–123.
- [22] P. SCHAUENBURG. *Hopf Bigalois extensions*. Comm. in Algebra **24** (1996) 3797–3825.
- [23] S. SKRYABIN. *Projectivity and freeness over comodule algebras*. Trans. Am. Math. Soc. 359 **6** (2007) 2597–2623.
- [24] K. -H. ULBRICH. *Galois extensions as functors of comodules*. Manuscr. Math. **59**, Number 4 (1987), 391–397.